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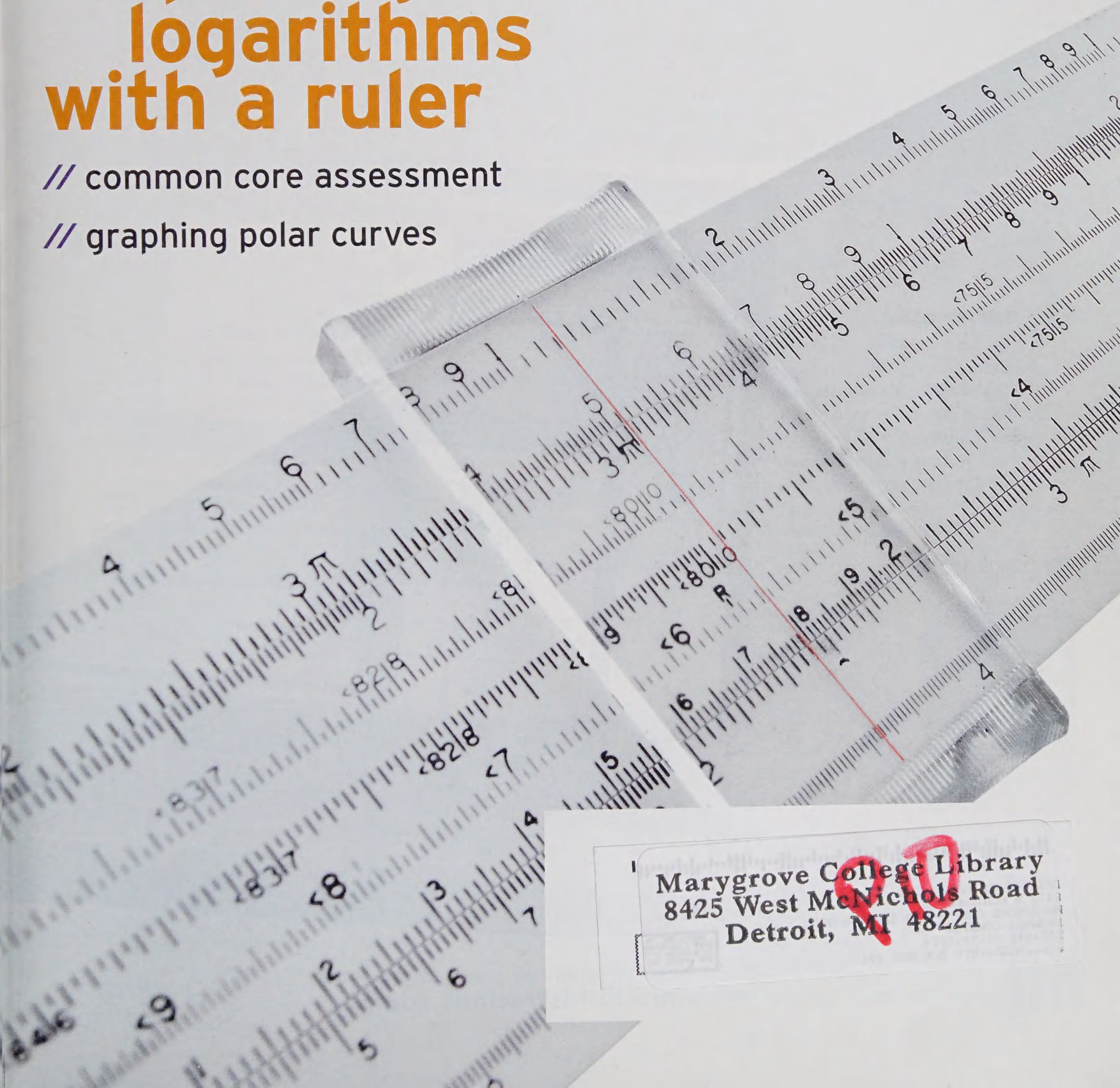
MATHEMATICS teacher

MAY 2013

exploring logarithms with a ruler

// common core assessment

// graphing polar curves



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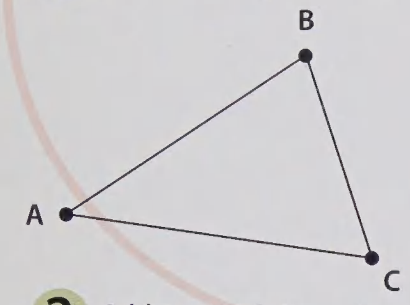

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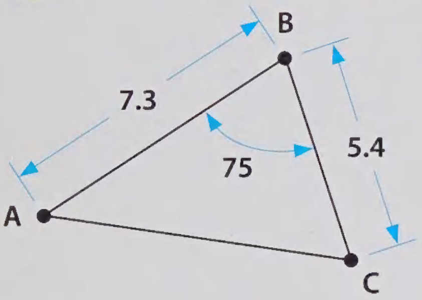

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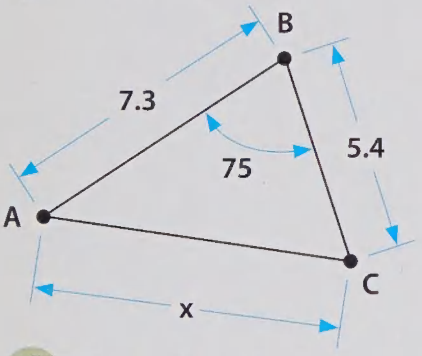

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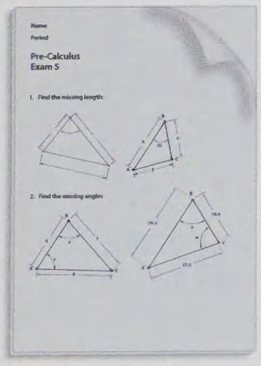

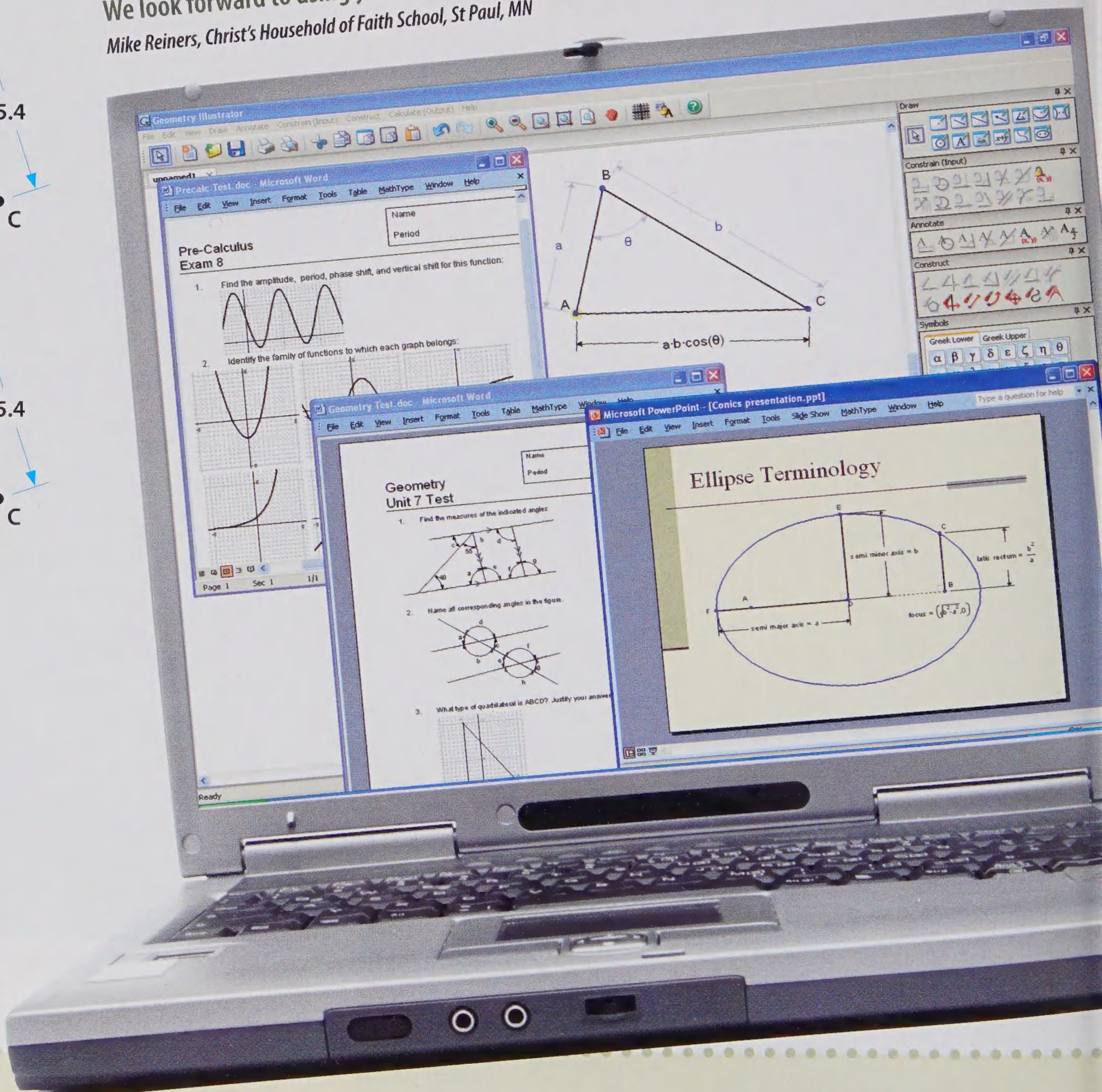
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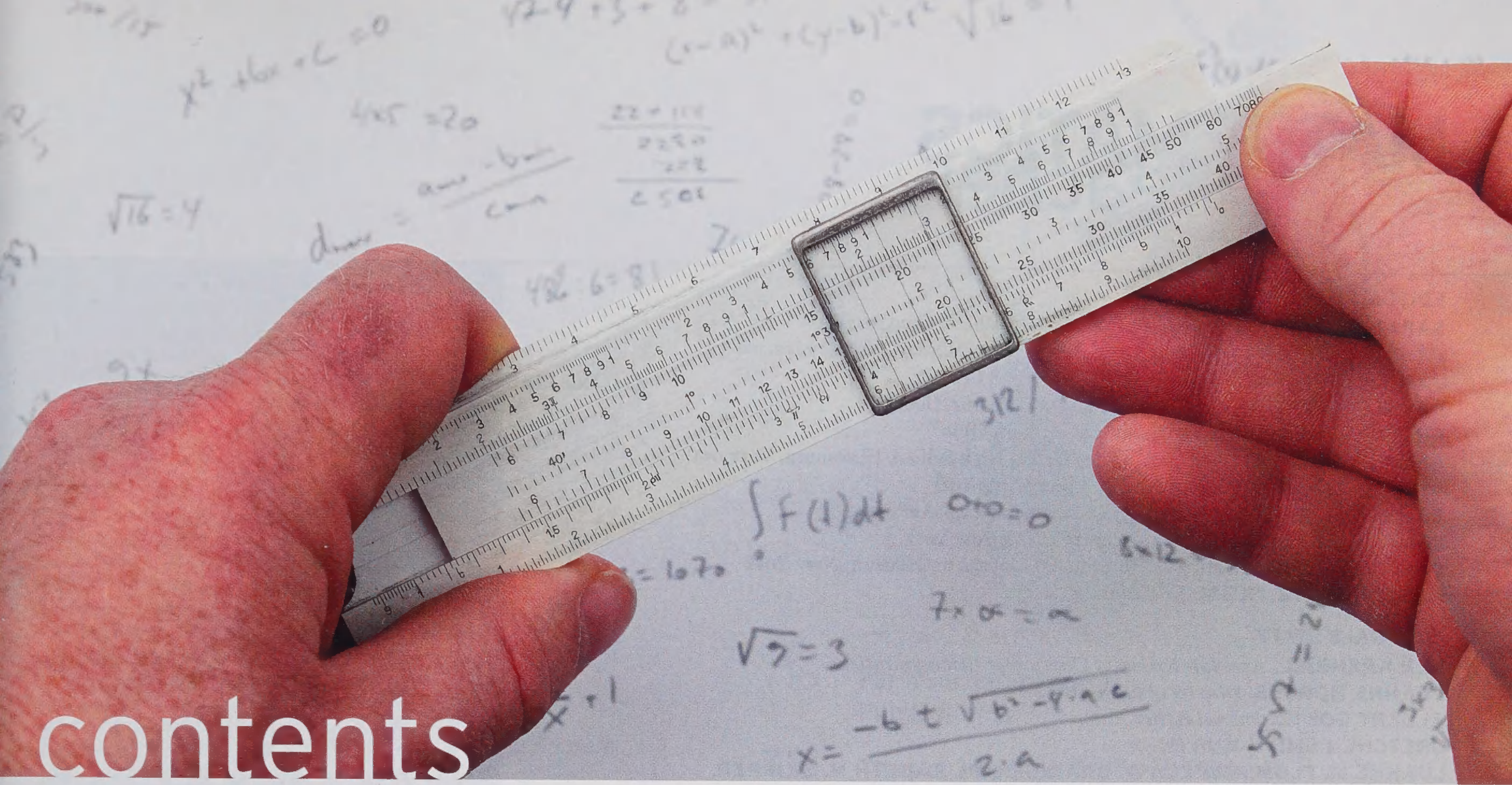

contents

Volume 106, Number 9, May 2013

features

656 **Prepare for More Realistic Test Results**
Matthew R. Larson and Steven Leinwand
The potential drop in reported state proficiency rates when the new CCSSM assessments are implemented will require adjusted expectations.

660 **Graphing Polar Curves**
Jonathan F. Lawes
Graphing the polar function on a rectangular plane simplifies graphing, increases student understanding, and reinforces connections.

668 **Exploring Logarithms with a Ruler**
Elliott Ostler
Processes using linear measurement can be adapted to teach complex topics such as polynomial multiplication, rational exponents, and logarithms.

674 **Counterexamples as Starting Points for Reasoning and Sense Making**
David A. Yopp
Asked to "fix" a false conjecture, students combine their reasoning and observations about absolute value inequalities, signed numbers, and distance to write true mathematical statements.

686 **Derivative of Area Equals Perimeter–Coincidence or Rule?**
Rina Zazkis, Ilya Sinitsky, and Roza Leikin
A familiar relationship–the derivative of the area of a circle equals its circumference–is extended to other shapes and solids.

on the cover

When did you last hear the word *nomograph*? Slide rules were long ago replaced by scientific and graphing calculators, but Elliot Ostler uses parallel number lines—a nomograph—to help students gain insight into the laws of exponents and how logarithms work. Turn to page 668 and read "Exploring Logarithms with a Ruler."

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Mathematics Teacher, an official NCTM journal, is devoted to improving mathematics instruction for grades 8–14 and supporting teacher education programs. It provides a forum for sharing activities and pedagogical strategies, deepening understanding of mathematical ideas, and linking mathematics education research to practice. *Mathematics Teacher* solicits submissions from high school mathematics teachers, university mathematicians, and mathematics educators and strongly encourages manuscripts in which ideas relate to classroom practice. Manuscripts previously published in other journals or books are not acceptable. NCTM publications present a variety of viewpoints. The views expressed or implied in this journal, unless otherwise noted, should not be interpreted as official NCTM positions. The appearance of advertising in NCTM's publications and on its websites in no way implies endorsement of approval by NCTM of any advertising claims or of the advertiser, its product, or its services. NCTM disclaims any liability whatsoever in connection with advertising appearing in NCTM's publications and on its websites.

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departments

644 **Reader Survey of NCTM Journals (2011)**

645 **Reader Reflections**

648 **Media Clips**
World Population Soon to Hit 7 Billion // Facebook Mathematics
Ryan Brydges/Kevin Webster
Edited by Louis Lim and Lionel Garrison

652 **Mathematical Lens**
Going with the Grain
Edited by Ron Lancaster and Brigitte Bentele

680 **June 2013 Calendar and Solutions**
Edited by Margaret Coffey and Art Kalish

694 **Activities for Students**
Connecting Probability and Algebra
Flor Jacqueline Alarcón Mejía
Edited by Ruth Dover and Patrick Harless

704 **Connecting Research to Teaching**
Reasoning about Quantities That Change Together
Heather Lynn Johnson
Edited by Margaret Kinzel and Laurie Cavey

710 **Delving Deeper**
The Diver Problem: The Surfer Problem in 3D
Dean B. Priest, Ronald G. Smith, Christin Carlisle, and Rebecca Mays
Edited by Maurice Burke, J. Kevin Colligan, Maria Fung, and Jeffrey J. Wanko

710

716 **For Your Information**
Publications

718 **Thanks from the Editorial Panel**

720 **The Back Page: My Favorite Lesson**
The Bulldog Lottery
Kenneth A. Horwitz
Edited by Jennifer Wexler

calls for manuscripts

683 **Connecting Research to Teaching**

691 **Technology Tips**

693 **Open the Door and Keep It Open**

703 **2015 Focus Issue: Creating Classroom Communities**

715 **On the Front Burner: Emerging Issues in Mathematics Education**

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Look online for these additional items:

- Mathematical Lens (p. 652): An additional photograph for your own mathematical questions
- Activities for Students (p. 694): Customizable activity sheets

Download one of the free apps for your smartphone. Then scan this tag to access **www.nctm.org/mt.**

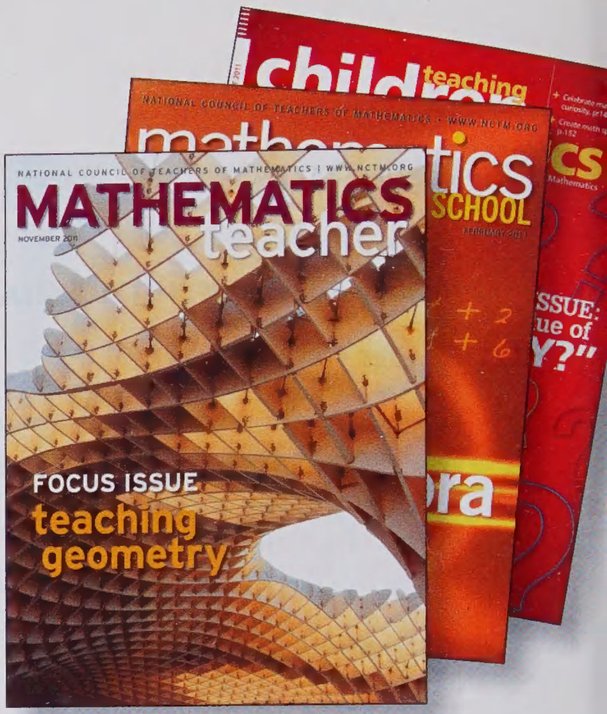
in the next issue

Coming in August 2013:

- "Modeling Fuel Efficiency: Why the EPA Is Changing from MPG to GPHM," by Kevin G. Bartkovich
- "Improving Student Reasoning with Parallel Tasks," by Bobson Wong and Larisa Bukalov
- "Applications in Digital Image Processing," by Jason Silverman, Gail L. Rosen, and Steve Essinger

READER SURVEY

OF NCTM JOURNALS (2011)



The authors, editors, Editorial Panel members, and staff who work on the NCTM school journals—*Teaching Children Mathematics*, *Mathematics Teaching in the Middle School*, and *Mathematics Teacher*—continually strive to meet readers' needs. To determine how well we all succeed, we gather feedback every three years through a reader survey and accompanying focus groups. Two such inquiries have been performed to date by an outside consulting firm, Stratton Publishing & Marketing/Stratton Research. For the most recent (2011) survey, we invited 3000 readers of each journal to participate; the effective mailing was 8457. For all three journals, the response rate averaged 26.4%, which is higher than for many other journals. Responses were calculated at the 95% confidence level with the average margin of error around ±3.5%.

Of the twenty-five questions posed, some of the most revealing asked whether the NCTM journals—

- are an effective resource for practicing and prospective teachers and supervisors;
- are useful on the job;
- have content that is credible, timely, relevant, and well written; and
- are designed to communicate the content effectively.

The level of positive response was far above what is considered the industry standard. In fact, it so exceeded the standard that Stratton Research, in a recent issue of its newsletter for association and university clients, featured the NCTM journals as an example of excellence.

Although gratified by the positive input, the *MT* Editorial Panel and staff also took note of several strands of critical comments in the anecdotal responses. Readers who complained fell into three large groups:

- Those who felt that the level of the journal was too high for the caliber of students with whom they work
- Those who wanted more articles written by high school teachers
- Those who wanted more ready-to-use ideas for the classroom

The *MT* Editorial Panel has been aware of these concerns for some time and has made recent efforts to respond to them. For example, the theme of the 2011 and 2013 Focus Issues was the mathematical foundations of, respectively, geometry and algebra. The Back Page: My Favorite Lesson, which is intended to provide a forum for high school teachers, has provided many classroom-ready lesson ideas. And *MT*'s call for manuscripts on entry-level classes, which has recently morphed into the call Open the Door and Keep It Open (see p. 709), is designed to encourage submissions of just the kind of journal articles that readers say they want.

But we can publish only what is submitted. We have heard from high school teachers that they do not have time to write or that they do not write well, so we are looking into ways to alleviate those concerns, such as pairing classroom teachers with university teacher-educators. We hope that high school teachers who have good ideas that they have put into practice in their own classrooms will submit manuscripts—either in response to a call or not—soon and often.

READER reflections

HOW MANY JELLY BEANS?

The Common Core Mathematics Standards: Transforming Practice through Team Leadership (NCTM and Corwin 2012), by Ted H. Hull, Ruth Harbin Miles, and Don S. Balka, is an excellent resource. The book's appendix contains some sample problems that show CCSSM content and practices.

One problem for grade 7 (p. 137) is given as follows:

> Three students guessed the number of jelly beans in a jar. Their guesses were 348, 359, and 368. The guesses were off by 1, 12, and 8 but not in the order of their guesses. How many jelly beans could be in the jar?

The discussion of the problem states that there are two possible solutions: 352 jelly beans and 360 jelly beans. Unfortunately, this statement is wrong; only one solution—360 jelly beans—is correct. Realizing this error led me to wonder whether there are similar problems that have two distinct solutions.

Suppose that the two correct answers were 352 and 360. What guesses could have produced these answers? To identify these guesses, I first considered the three errors: A, B, and C.

One of the guesses could be halfway between the two correct answers. This situation is modeled by the following equations:

$$352 + A = \text{guess } 1 = 360 + B \rightarrow$$
$$A = 8 + B \rightarrow A - B = 8$$

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged, but because of the large number submitted, we do not send letters of acceptance or rejection. Letters to be considered for publication should be in MS Word document format and sent to **mt@nctm.org**. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including e-mail address, per the style of the section.

THE DISCUSSION continues

Following is a response to Linda M. Gojak's reply to the question "Is NCTM Political Enough?" which appeared in Reader Reflections in *MT* Dec. 2012/Jan. 2013 (vol. 106, no. 5, pp. 324–25). Ms. Gojak is the president (2012–14) of NCTM.

WE NEED YOUR HELP, NCTM

Having been invited to respond to "Is NCTM Political Enough?" I offer the following: NCTM is not.

Although President Gojak lists various ways in which NCTM has spoken out, the organization has been largely unheard and mostly ineffective. In her first paragraph, Ms. Gojak admits that NCTM's work is "not widely publicized." In the second paragraph, she mentions formative assessment. Formative assessment is not the issue here; it may be important as a pedagogical technique but not as a political issue.

The political issues revolve around the fact that most administrators on the high school level (with which I am most familiar) are ill equipped to assess quality mathematics teaching. They have little knowledge of the content to begin with and even less of constructivist pedagogy. As a result—and I have experienced this firsthand—such administrators criticize when a standard is not displayed at the front of the classroom and side with those students and parents who complain that "the teacher isn't teaching the steps."

In their wonderful article "Authentic Tasks in a Standards-Based World" (*MT* Dec. 2012/Jan. 2013, pp. 346–53), Edwards, Harper, and Cox state that the Fermat point is not in the Ohio Standards and that "teachers won't teach content like this unless it's in the Content Standards." Wow! What a loss of a terrific lesson, if in fact this is so.

We need your help, NCTM. Administrators will not accept the opinion of an individual teacher as to the nature of discovery teaching of mathematics nor allow for flexibility as to how a lesson develops (a la the Fermat point case) and the extension, beyond written standards, of various mathematical ideas.

A few years ago, I was criticized for not displaying a standard on the chalkboard for students to see. The next time an administrator visited my class, I was complimented for posting a standard. I had deliberately put up a standard that was not at all related to the mathematics we were doing that day. No student mentioned it, either. Has anyone determined that students learn better when the precisely worded standard is displayed somewhere?

Finally, the most troublesome issue to me is the "one-size-fits-all" aspect of standardized testing. Is it reasonable to assume that special-needs students, with their many varied disabilities, will all perform at the same absolute level as students without such learning disabilities? Should it not be our goal to assist all students to reach their maximum potential without comparing that maximum with those of other students? We need your help, NCTM.

NCTM's goal has always been, quoting Ms. Gojak, "to reflect progressive, effective practices and to inform and inspire classroom teachers." We need NCTM's help so that we can reach a wider audience. We *need* NCTM to be more political.

Ken McCaffrey
Kenneth.mccaffrey@yahoo.com
Brattleboro Union High School (retired)
Brattleboro, VT, Dec. 12, 2013

$$352 - B = \text{guess } 2 = 360 - A \rightarrow$$
$$-B = 8 - A \rightarrow A - B = 8$$

$$352 + C = \text{guess } 3 = 360 - C \rightarrow$$
$$2C = 8 \rightarrow C = 4$$

This problem, therefore, has a solution if one error is 4 and the difference between the other two errors is 8. The solution $A = 13$, $B = 5$, and $C = 4$, for example, would mean that the guesses are 365, 347, and 356.

These values lead to a revised version of the original problem:

> Three students guessed the number of jelly beans in a jar. Their guesses were 347, 356, and 365. The guesses were off by 4, 5, and 13 but not in the order of their guesses. How many jelly beans could be in the jar?

The solutions to this problem are 352 or 360: $347 + 5 = 352$, $356 - 4 = 352$, $365 - 13 = 352$; and $347 + 13 = 360$, $356 + 4 = 360$, $365 - 5 = 360$.

Another possibility might be that none of the guesses is halfway between the two correct answers. That case is modeled by the following equations:

$$352 + A = \text{guess } 1 = 360 + B \rightarrow$$
$$A = 8 + B$$

$$352 + C = \text{guess } 2 = 360 - B \rightarrow$$
$$C = 8 - B$$

$$352 - A = \text{guess } 3 = 360 - C \rightarrow$$
$$-A = 8 - C$$

Adding the three equations on the right yields $C = 24 - C \rightarrow C = 12$.

Two errors of the same magnitude, as is the case here, would change the essence of the problem; it would now state, "The guesses were off by 4, 4, and 12." Therefore, I ignored the case that none of the guesses is halfway between the two correct answers.

This method can be used to construct similar problems that have two solutions. All that is necessary is for the two solutions to have the same parity (both even or both odd).

John F. Mahoney
johnf.mahoney@gmail.com
Benjamin Banneker Academic High School
Washington, DC, July 29, 2012

PROBLEM 24, SEPTEMBER 2012 CALENDAR

Problem 24 of the September 2012 Calendar states (see *MT* August 2012, vol. 106, no. 1, p. 41):

> The integers 220, 251, and 304 represent three consecutive perfect squares in base b. Determine the value of b.

As with the printed solution, an alternate route would begin with the realization that consecutive squares differ by consecutive odd integers. Since the difference between 251 and 220 is 31 (in base b or base 10), the difference between the next two squares must be 33. Since 304 and 251 have a difference of 33 in base b, 25 must be 3 less than 30 in base b. Therefore, the necessary base is 8.

Thomas K. Rice
tkrice@tsc.k12.in.us
East Tipp Middle School
Lafayette, IN, Aug. 6, 2012

PROBLEM 31, OCTOBER 2012 CALENDAR

Problem 31 of the October 2012 Calendar (see *MT* September 2012, vol. 106, no. 2, p. 121) is stated as follows:

> Given the parametric equations
>
> $$x = 2a\tan\theta \text{ and } y = 2a\cos^2\theta,$$
>
> eliminate the parameter (θ) and solve the resulting equation for y in terms of a and x.

I suggest an alternate to the published solution. The parametric equations $x = 2a\tan\theta$ and $y = 2a\cos^2\theta$ can be rewritten as $\tan\theta = x/2a$ and as $\sec^2\theta = 2a/y$, respectively. The fact that $\tan^2\theta + 1 = \sec^2\theta$ gives

$$\left(\frac{x}{2a}\right)^2 + 1 = \frac{2a}{y},$$

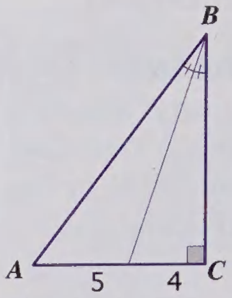
from which we can conclude that

$$y = \frac{8a^3}{x^2 + 4a^2}.$$

Steven Siegel
siegel443@comcast.net
Niagara University (retired)
Niagara, NY, Sept. 7, 2012

PROBLEM 26, NOVEMBER 2012 CALENDAR

Problem 26 of the November 2012 Calendar (see *MT* October 2012, vol. 106, no. 3, p. 201) is stated as follows:



> In right triangle ABC, the bisector of $\angle ABC$ divides the opposite leg into two segments of length 5 and 4. Find the area of $\triangle ABC$.

This is a great exercise to show that a problem involving similarity and a Pythagorean triple from a geometry course can be solved with a trigonometric identity from a traditional second-year algebra course.

Label the intersection of the angle bisector and AC as D and let $BC = x$. Then, using the double-angle formula, we have

$$\tan\angle ABC = \frac{2\tan\angle DBC}{1 - \tan^2\angle DBC}.$$

But $\tan\angle ABC = 9/x$, and $\tan\angle DBC = 4/x$. So

$$\tan\angle ABC = \frac{2 \bullet \frac{4}{x}}{1 - \frac{16}{x^2}} = \frac{8x}{x^2 - 16}.$$

Setting our two expressions for the tangent of angle ABC equal to each other, we obtain the following:

$$\frac{9}{x} = \frac{8x}{x^2 - 16} \rightarrow$$
$$9x^2 - 144 = 8x^2 \rightarrow$$
$$x = 12$$

The requested area of the triangle is thus 54.

Dick Smith
dsmith@dbq.edu
University of Dubuque
Dubuque, IA, Oct. 9, 2012

Editor's note: A similar letter was received from Steven Siegel, Niagara University (retired).

NEGATIVE TIME

A standard application of quadratic functions asks students to determine the time when an object propelled from a height above the ground will reach the ground. For example, the equation $d = -16t^2 + 32t + 48$ gives the position d (in feet above ground) at time t (in seconds) of an object launched with an initial velocity of 32 ft./sec. from a height of 48 feet above ground. To determine when the object will hit the ground, we replace d with 0 and solve the resulting quadratic, obtaining $t = -1$ or $t = 3$. We announce that the object hits the ground after 3 seconds, rejecting the negative value for t. Perhaps we can make that value meaningful.

If we translate the graph of $d = -16t^2 + 32t + 48$ one unit to the right, we have $d = -16(t - 1)^2 + 32(t - 1) + 48$, which simplifies to $d = -16t^2 + 64t$. This equation describes the position of an object launched from the ground with an initial velocity of 64 ft./sec., reaching a height of 48 feet one second later, a height of 64 feet after another second, returning to 48 feet after another second, and hitting the ground after another second. This is the same flight pattern as in the first case, except that the object uses the first second to get to the height of 48 feet. Thus, we could have launched the object from the ground one second earlier than we launched it from the platform, using an initial velocity of 64 ft./sec., and the object would have traveled the same route in the remaining 3 seconds that it followed when launched from a height of 48 feet.

We could also apply physics to the situation and determine that the object was traveling at 64 ft./sec. when it reached a height of 48 feet on its descent, since $dy/dt = -32(2) = -64$. Thus, by symmetry the initial velocity needed to propel the object from the ground to a height of 48 feet would be 64 ft./sec. The corresponding equation would be $d = -16t^2 + 64t$, as before.

James Metz
metz@hawaii.edu
Kapi'olani Community College
Honolulu, HI, May 12, 2012

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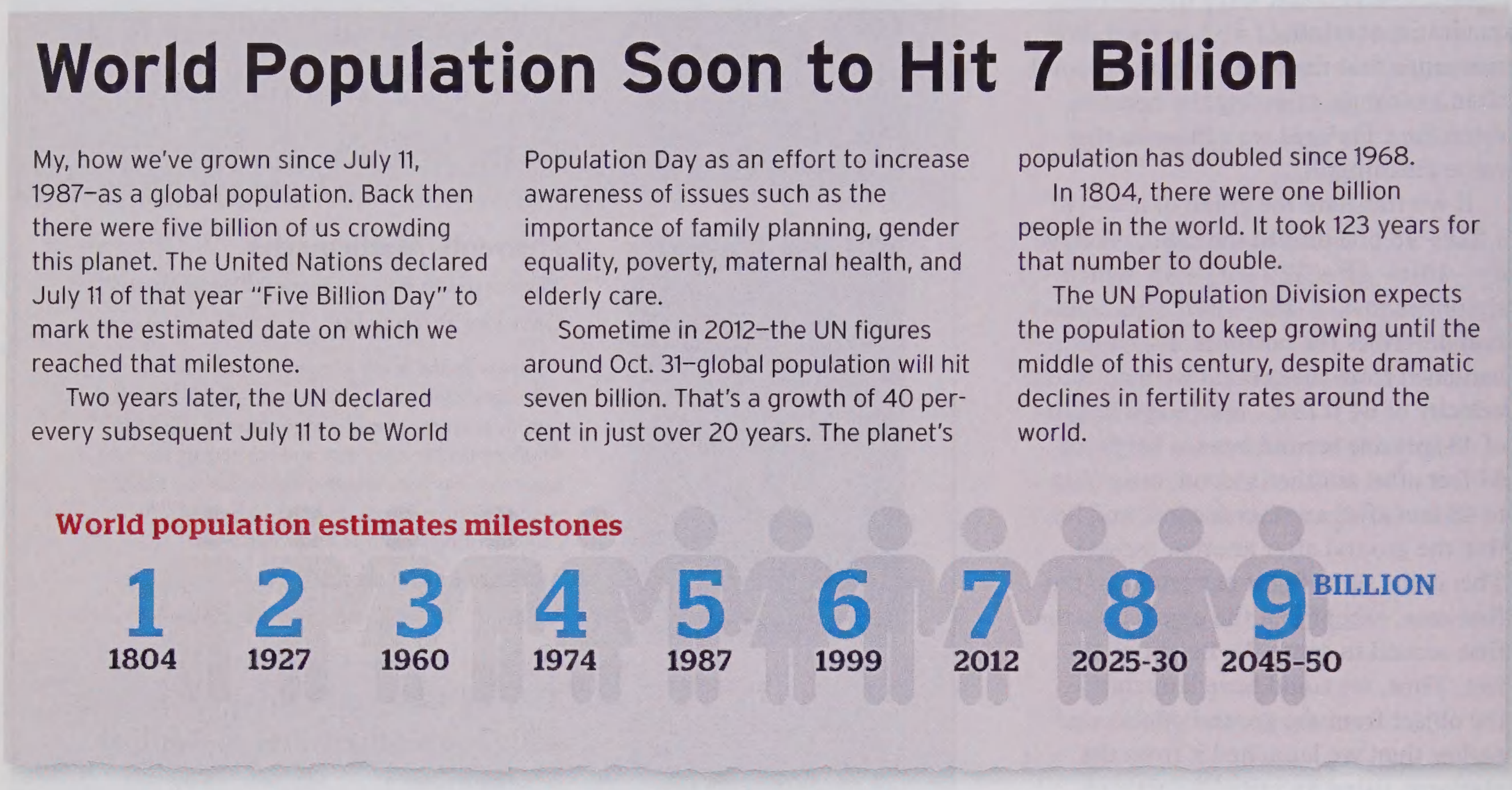

World Population Soon to Hit 7 Billion

My, how we've grown since July 11, 1987–as a global population. Back then there were five billion of us crowding this planet. The United Nations declared July 11 of that year "Five Billion Day" to mark the estimated date on which we reached that milestone.

Two years later, the UN declared every subsequent July 11 to be World Population Day as an effort to increase awareness of issues such as the importance of family planning, gender equality, poverty, maternal health, and elderly care.

Sometime in 2012–the UN figures around Oct. 31–global population will hit seven billion. That's a growth of 40 percent in just over 20 years. The planet's population has doubled since 1968.

In 1804, there were one billion people in the world. It took 123 years for that number to double.

The UN Population Division expects the population to keep growing until the middle of this century, despite dramatic declines in fertility rates around the world.

World population estimates milestones

BILLION	1	2	3	4	5	6	7	8	9
	1804	1927	1960	1974	1987	1999	2012	2025-30	2045-50

Source: http://www.cbc.ca/news/world/story/2011/07/10/f-world-population-day-7billion.html

1. Suppose that, as the article states, the world's population was 5 billion in 1987 and will be 7 billion in 2012.
   - (*a*) Determine the average annual growth rate.
   - (*b*) Write an equation to model this growth. Explain why you chose this model.
2. Use your model to calculate—
   - (*a*) the world population in 1960; and
   - (*b*) the world population in 2025.
3. How do your predictions compare with the information in the article about the actual population in 1960 and the predicted population in 2025? What does this say about the growth rate that you calculated in question 1?
4. (*a*) What do you predict the population growth will be in 2025?
   - (*b*) Do you think that your model will accurately predict these values? Why or why not?
5. (*a*) Graph the data from the article for the years from 1804 through 2045–50. Use time for the horizontal axis and population for the vertical axis.
   - (*b*) Describe the rate of change on each part of the graph.
6. (*a*) Use a graphing calculator to determine the equation of a logistic model for the data.
   - (*b*) Compare the graph of the logistic model with the graph of the model that you created for question 1. Which model better represents this scenario?

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Facebook Mathematics

MAX PHOTOGRAPHY/ISTOCKPHOTO

The warning came in an e-mail one morning last November: "Crystal Kapteyn, you have too many friends." The avid Facebooker had unknowingly surpassed the social network's friends limit. Her number? A whopping 5,000.

"I just kept accepting people and accepting people. And then it got out of control," says Kapteyn, a mother and a political science student at Brock University. Moderators suggested she join the likes of Justin Bieber and Prime Minister Stephen Harper by creating a Facebook fan page. "I was like, 'A fan page? I'm just a normal person!'"

The only other option was to do a mass purge. Kapteyn, 32, took a deep breath and posted a status update: "Starting deletions." It took nearly seven hours of clicking and scrolling—"It was kind of ridiculous, to be honest," she says—but Kapteyn did it. Today, her number is 1,578, heaps less than before but still a far cry from the average Facebook user's 120 friends.

Even with 3,500 (approximately) fewer friends than she used to have, Crystal Kapteyn gets thousands of Facebook notifications sent to her BlackBerry and e-mail every month.

Source: Amy Dempsey, "When Even Facebook Thinks You Have Too Many Friends," *Toronto Star*, June 30, 2011, http://www.thestar.com/life/2011/06/30/when-even-facebook-thinks-you-have-too-many-friends.html

1. Suppose that Facebook did not have a 5,000-friend limit. Presumably, the more people you friend, the more suggestions for friends you get; thus, your list may grow exponentially. Assuming that Crystal still has 5,000 friends and assuming that her list grows by 2.5% per month, answer the following questions:

   (*a*) Write an exponential equation that models the growth in the number of Crystal's friends.

   (*b*) How many friends will Crystal have 12 months from now?

   (*c*) In how many months will the number of Crystal's friends be double what it is now?

2. In cutting down from 5,000 to 1,578 friends in 7 hours, how many friends did Crystal delete on average—

   (*a*) in each hour?

   (*b*) in each minute?

3. If Crystal went through her entire list of friends (5,000) one at a time in the 7 hours, for how long (in seconds) did she consider each friend (on average) before keeping or deleting each one? Assume that Crystal takes a 30-minute lunch break and two 10-minute bathroom breaks during this process.

4. The author determined that there are 73 possible scenarios in which Facebook could send a user an e-mail notification, including anything from the user being tagged in a photograph to someone commenting on the user's wall. Therefore, managing these notifications is critical; otherwise, one's e-mail box may fill up quickly.

   Suppose that Crystal had selected all the e-mail notifications possible and that, by some strange coincidence one day, each of her 5,000 friends activated all 73 scenarios (they all tagged her in a photo, wrote on her wall, etc.). How many e-mail notifications would Crystal receive that day if it were—

   (*a*) before her deletions?

   (*b*) after her deletions?

5. Let's consider a more realistic scenario. On one particular day, she happened to get 1 e-mail notification (of any type) from 27% of her 5,000 friends.

   (*a*) What would be the total number of e-mail notifications that Crystal received?

   (*b*) If it takes Crystal 1 minute (on average) to read and act on each e-mail, how many minutes does she spend that day dealing with her Facebook e-mails? Do you see a problem with this?

   (*c*) Let's say that Crystal works for 18 hours per day on e-mails (after all, she has to sleep and eat). How many minutes on average would she fall behind each day?

MEDIA CLIPS solutions

"World Population" answers

1. (*a*) We can calculate an annual growth rate in several ways. For instance, the world population grew from 5 billion in 1987 to 7 billion in 2012. This is an increase of 2 billion over 25 years, an average increase of 0.08 billion people per year. An alternative way to analyze the data is to recognize that the change from 5 to 7 billion is a percent increase of 40%, or 1.6% per year.

   (*b*) We can determine a model for the population growth described in the article based on each assumption in (*a*).

   A linear model would use the increase of 0.08 billion people per year as the slope in the equation $P = 0.08n + 5$ where P represents population in billions, n is the number of years since 1987, and 5 is the population in 1987 ($n = 0$) in billions.

   An exponential model would use the average annual increase of 1.6% to arrive at the equation $P = 5(1.016)^n$ where P represents population in billions and n represents the number of years since 1987.

   Students comfortable with the concept of continuous growth might solve the equation $7 = 5e^{25r}$ for r, the rate of increase over the 25 years from 1987 to 2012. When that equation is solved for r, we obtain

$$r = \frac{\ln \frac{7}{5}}{25} \approx 0.01346,$$

   yielding a different exponential equation of $P = 5e^{0.01346n}$.

2. (*a*) Each model gives a different estimate of the population in 1960. If we use $P = 5(1.016)^n$, $n = -27$, we get $P = 5(1.016)^{-27} \approx 3.257$ billion. Using $P = 5e^{0.01346n}$ with $n = -27$ yields an estimate of 3.476 billion in 1960. If we use the linear model, $P = 0.08n + 5$ with $n = -27$, we get an estimate of 2.84 billion.

   (*b*) Predicting the population in 2025 ($n = 38$) will also vary by model:

$$P = 5(1.016)^{38} \approx 9.140 \text{ billion}$$
$$P = 5e^{38(0.01346)} \approx 8.339 \text{ billion}$$
$$P = 5 + 0.08(38) = 8.04 \text{ billion}$$

3. The actual population in 1960 was about 3 billion. The predicted population as stated in the article for the years 2025–30 is 8 billion.

   When rounded to the nearest billion people, all three models predict the population in 1960 very well. The models diverge much more in predicting the population in 2025. The first exponential model, using a percent increase of 1.6%, is larger by 1 billion people, suggesting that 1.6% is too high a growth rate for these later years.

4. (*a*) Answers will vary but might indicate that the population will continue to increase, although the rate of increase will fall until the population levels off at around 10 billion.

   (*b*) The results from question 3 suggest that the population growth rates have not remained constant. We would expect the predictions from the models to become less accurate as we predict values further and further in the future.

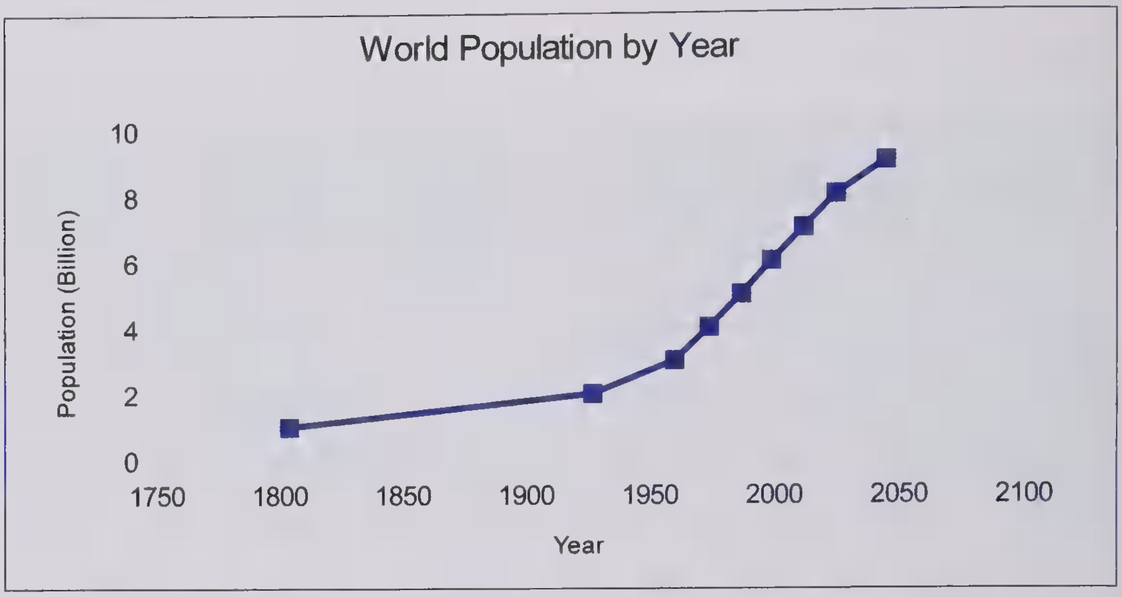


Fig. 1 ("World Population")

Table 1 ("World Population")

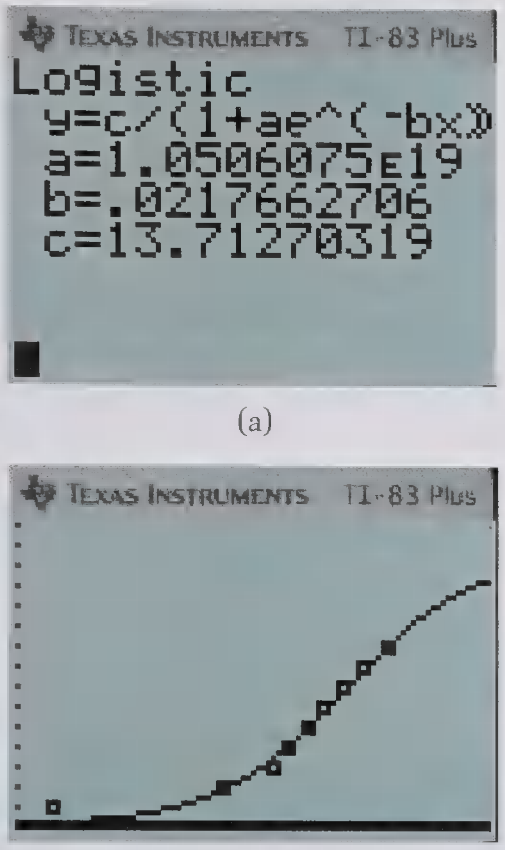
Time Interval	Rate of Change (Billions/Year)
1804–1927	0.008
1927–1960	0.03
1960–2025	0.08
2025–2045	0.05

5. (*a*) See **figure 1 ("World Population")**.

   (*b*) See **table 1 ("World Population")**. The growth rate starts small and then grows to reach a maximum somewhere in the interval between 1960 and 2025. It then begins to slow down in the years following 2025.

6. (*a*) The logistic curve in **figure 2 ("World Population")** fits the data very well, much better than either the exponential or the linear model.

   (*b*) The logistic model better represents this scenario. For both an exponential and a linear model, as x gets larger, y also gets larger.



```
Logistic
 y=c/(1+ae^(-bx))
 a=1.0506075E19
 b=.0217662706
 c=13.71270319
```

(a)

(b)

Fig. 2 ("World Population")

Since we know that a population cannot experience unbounded growth and the logistic model levels off at close to 14 billion people, it is the best model for these data.

"Facebook Mathematics" answers

1. (*a*) Let $F(n)$ represent the number of Crystal's friends and let n represent the number of months. We know that the growth will be exponential, so we can model the situation as follows:

$$F(n) = 5{,}000(1.025)^n$$

or, as a continuous growth function,

$$F(n) = 5{,}000e^{0.025n}.$$

(*b*) Evaluate $F(n)$ when $n = 12$ (12 months from now):

$$F(n) = 5{,}000(1.025)^{12} \approx 6{,}724$$

or

$$F(n) = 5{,}000e^{0.3} \approx 6{,}749.$$

Therefore, Crystal will have approximately 6,724 or 6,749 friends 12 months from now, depending on which growth model we use.

(*c*) Find n when $F(n) = 10{,}000$ (when Crystal has doubled her current number of friends). We use $F(n) = 5{,}000\ (1.025)^n$. The analysis is similar for the continuous growth model:

$$\begin{aligned} 10{,}000 &= 5{,}000(1.025)^n \rightarrow \\ 2 &= 1.025^n \rightarrow \\ \log 2 &= n \log 1.025 \rightarrow \\ n &\approx 28.07 \end{aligned}$$

Therefore, Crystal will have double the number of her current friends (achieving 10,000 friends) in about 2 years and 4 months (28 months) from today.

2. We compute the number deleted: 5,000 – 1,578 = 3,422.

(*a*) The deletion rate per hour equals

$$\frac{\text{number deleted}}{\text{hr. spent working}} = \frac{3{,}422}{7} \approx 489 \text{ friends/hr.}$$

(*b*) The deletion rate per minute equals

$$\frac{\text{number deleted}}{\text{min. spent working}} = \frac{3{,}422}{7 \cdot 60} \approx 8 \text{ friends/min.}$$

Therefore, Crystal's deletion rate is about 489 friends per hour or about 8 friends per minute.

3. Crystal would have to look at each of her 5,000 friends as she considered whether to keep or reject each. The total time available is 7 hours minus the 50 minutes of breaks—that is, 6 hours and 10 minutes—which equals 370 minutes, or 22,200 seconds. The average time spent per friend is

$$\frac{22{,}200 \text{ sec.}}{5{,}000 \text{ friends}} = 4.44 \text{ sec./friend.}$$

Therefore, Crystal considers each friend for about 4.44 seconds before accepting or rejecting each.

4. (*a*) Before deletions:

$$\begin{aligned} &5{,}000 \text{ friends} \cdot 73 \text{ e-mails/friend} \\ &= 365{,}000 \text{ e-mails} \end{aligned}$$

(*b*) After deletions:

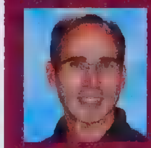
$$\begin{aligned} &1{,}578 \text{ friends} \cdot 73 \text{ e-mails/friend} \\ &= 115{,}194 \text{ e-mails} \end{aligned}$$

Therefore, on that day Crystal would have received 365,000 e-mails if she had 5,000 friends or 115,194 e-mails if she had 1,578 friends (either of which would fill up any e-mail box).

5. (*a*) 5,000 friends · 0.27 = 1,350 e-mails. Therefore, she would have received 1,350 e-mails that day.

(*b*) If it takes Crystal 1 minute to review each e-mail, then it will take her 1,350 minutes to review all the e-mails. The problem with this approach is that it does not consider time spent eating, sleeping, and doing other vital activities during the 1,440 minutes of each day. So, realistically, Crystal does not have time to deal with all the e-mails.

(*c*) If Crystal works for 18 hours per day on e-mails, she would have 18 hr./day · 60 min./hr. = 1,080 min./day to work on e-mails. Therefore, she would fall behind by 1,350 min. – 1,080 min. = 270 min./day.



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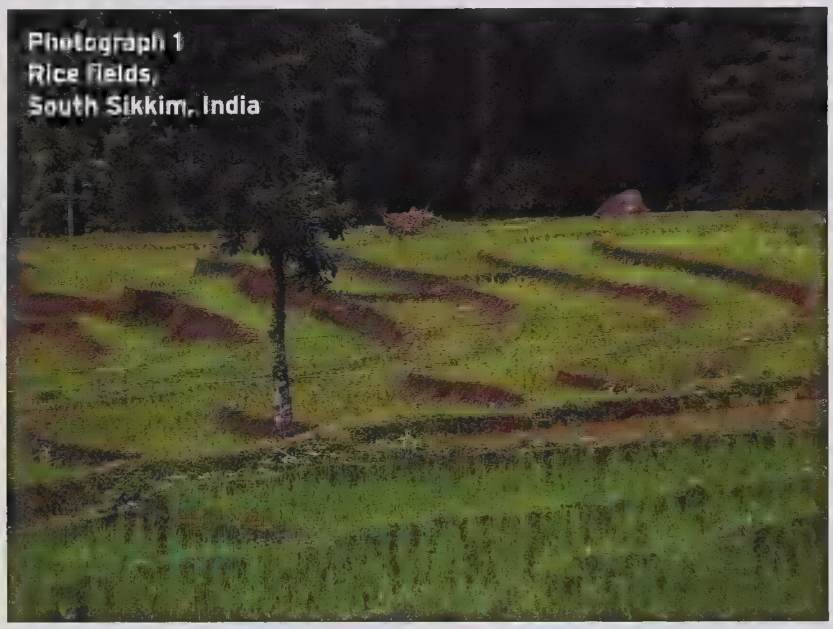
Going with the Grain



Photograph 1
Rice fields,
South Sikkim, India

BRIGITTE BENTELE

Photograph 2
Temi Tea Plantation,
South Sikkim, India

BRIGITTE BENTELE

Rice farmers in South Sikkim, India, might not think about sinusoidal functions when they plant and work in their paddies, but the beautiful patterns of the rice terraces on the Himalayan mountainsides recall the shape of trigonometric graphs (see **photograph 1**). The paths of the tea plantation seem to follow sinusoidal curves as well (see **photograph 2**).

Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors. For more background information on Mathematical Lens and guidelines for submitting a photograph and questions, please visit http://www.nctm.org/publications/content.aspx?id=10440#lens.

*Edited by **Ron Lancaster,** ron2718@nas.net, University of Toronto, ON, Canada, and **Brigitte Bentele,** brigitte.bentele@trinityschoolnyc.org, Trinity School, New York, NY*

1. The Geometer's Sketchpad® version of the rice fields in **photograph 1** shows five sinusoidal functions (see **fig. 1**).

   (*a*) Discuss the similarities and the differences between the functions.

   (*b*) What transformation—translation, rotation, dilation, or reflection—must be performed on $f(x)$ to obtain one of the other curves?

   (*c*) Determine an equation for each function.

2. **Figure 2** shows the graphs of $y = \sin x$, $y = \cos x$, and $y = \sin x + \cos x$. Notice that all three curves are sinusoidal functions; for example, $y = \sin x + \cos x$ can be written as $y = A\sin(Bx + C)$. Find A, B, and C.

3. The sum of a sine and a cosine function of the same period is itself a sinusoid. In the questions that follow, you will explore the possibilities when the periods of the two functions are different.

   (*a*) Consider the function $y = \sin x + \cos(2x)$. Is the graph sinusoidal? Is it periodic?

   (*b*) What is the period of $y = \sin(2x) + \cos(3x)$?

   (*c*) What is the period of $y = \sin(x/2) + \cos(x/3)$?

   (*d*) In general, what is the period of the function $y = \sin(Bx) + \cos(Cx)$ where B and C are both positive rational numbers?

(*e*) Now consider the function $y = \sin x + \cos(2\pi x)$. Is it periodic? Why or why not?

4. The sine and cosine curves shown in **figure 1** have sinusoidal axes that are parallel to the x-axis. A constant added or subtracted from a sinusoidal function causes its graph to move up or down, respectively, while remaining parallel to the original sine curve. Consider what happens when this constant is replaced by a nontrigonometric function of x.

(*a*) Graph $y = x + \sin x$. Is this function periodic? Why or why not?

(*b*) Graph $y = |x| + \sin x$. Is this function symmetric with respect to the y-axis? Why or why not?

(*c*) Graph $y = 2^x + \sin x$. Describe and explain the behavior of the graph.

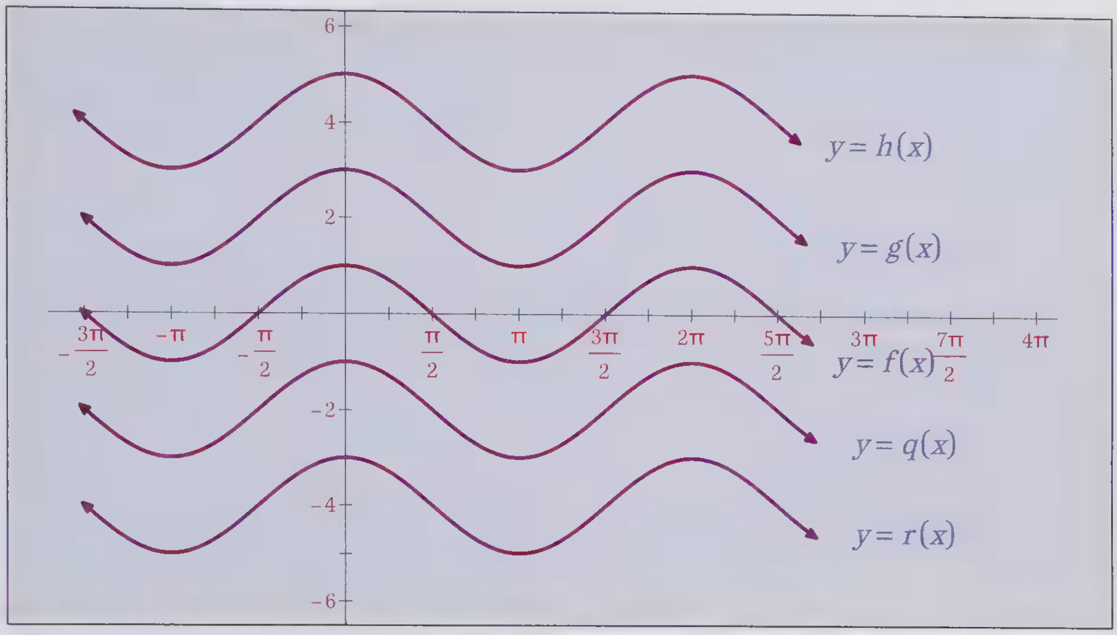


Fig. 1 Five sinusoidal functions can be entered in GSP.

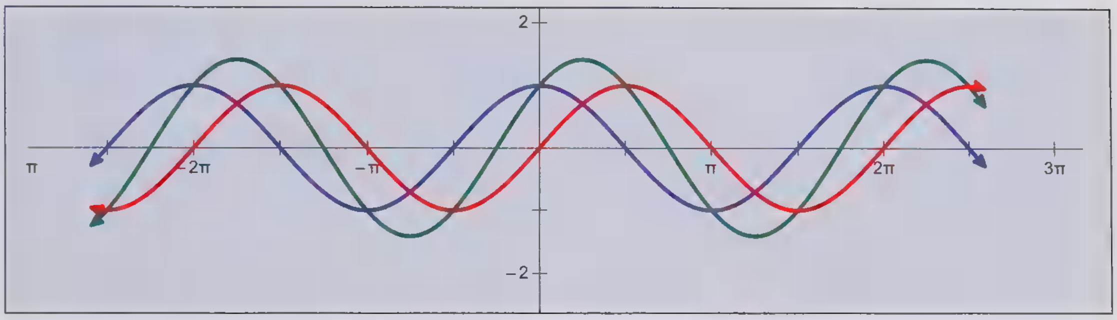
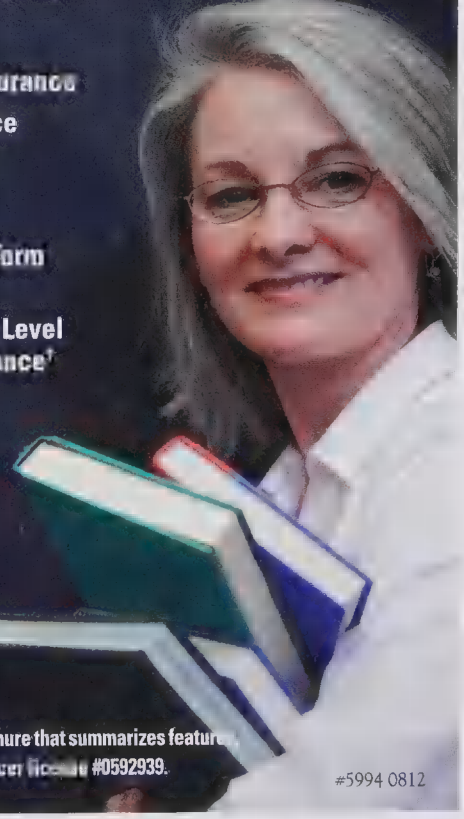


Fig. 2 The green curve, $y = \sin x + \cos x$, is the sum of the red and blue curves.



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MATHEMATICAL LENS solutions

1. (*a*) The graphs have the same amplitude, 1, and the same period, 2π, and are parallel to each other. However, the sinusoidal axis is different for each graph. (The *sinusoidal axis* is a line parallel to the x-axis midway between the high and low points of the sinusoid.)

   (*b*) The transformation is a translation of $f(x)$ either up or down.

   (*c*) There are many possibilities here, the most obvious of which are $h(x) = \cos x + 4$; $g(x) = \cos x + 2$; $f(x) = \cos x$; $q(x) = \cos x - 2$; and $r(x) = \cos x - 4$. These equations could also be written using sine instead of cosine, such as $f(x) = \sin(x + \pi/2)$.

2. We want $y = \sin x + \cos x$ to equal $y = A\sin(Bx + C)$. We first notice that the curves have the same period and therefore that $B = 1$. To find A and C, we use the identity for the sine of the sum of two angles: $y = A\sin(x + C) = A(\sin x \cos C + \cos x \sin C)$, which must equal $y = \sin x + \cos x$. We set the coefficients of $\sin x$ and $\cos x$ equal and then solve the simultaneous equations for A and C. Thus, $A\cos C = 1$, and $A\sin C = 1$. We divide the equations to obtain $A\sin C/(A\cos C) = 1$, making $\tan C = 1$ and therefore $C = \pi/4$. Substituting, we obtain

$$A = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \sqrt{2}.$$

   Therefore, the equation is

$$y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right).$$

3. (*a*) **Figure 3a** shows the graph of $y = \sin x + \cos(2x)$ on the interval $[-7\pi/2, 7\pi/2]$. A periodic function satisfies the equation $f(x) = f(x + p)$ where p is the period. This function is periodic, with a period of 2π.

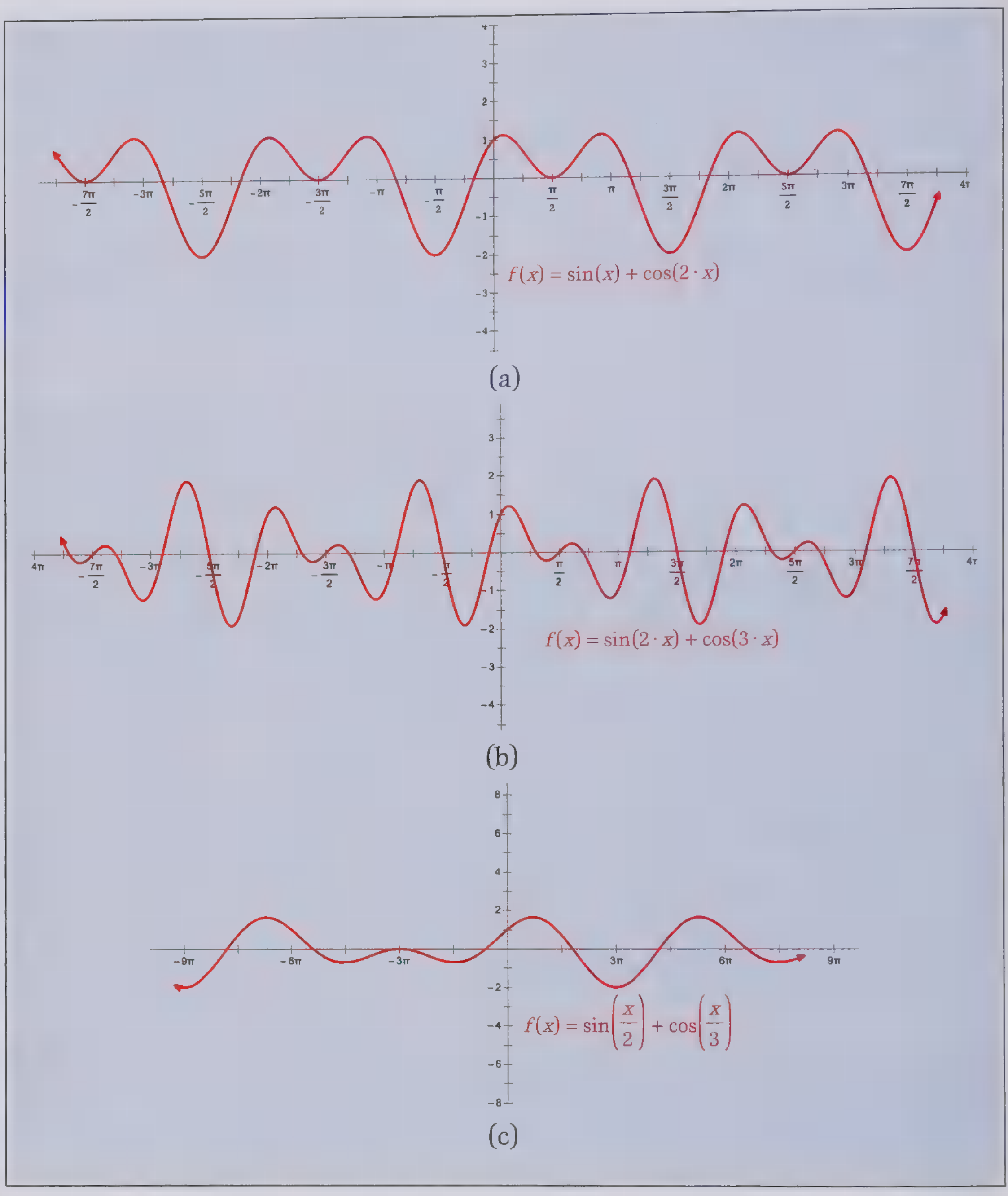

Fig. 3 One way to approximate a period is to look for matching x-intercepts. The graph in (**a**) has x-intercepts at $\pi/2$ and $5\pi/2$; in (**b**) at $\pi/2$ and $3\pi/2$; and in (**c**) at $-11\pi/2$ and $13\pi/2$.

   (*b*) **Figure 3b** shows the graph of $y = \sin(2x) + \cos(3x)$ on the interval $[-7\pi/2, 7\pi/2]$. Its period is 2π.

   (*c*) **Figure 3c** shows the graph of $y = \sin(x/2) + \cos(x/3)$ on the interval $[-9\pi, 9\pi]$. Its period is 12π.

   (*d*) The period of $y = \sin(Bx) + \cos(Cx)$ is the least common multiple of the period of each of the functions being added. The period of $y = \sin(Bx)$ is $2\pi/B$, and the period of $y = \cos(Cx)$ is $2\pi/C$. Then the period of $y = \sin(Bx) + \cos(Cx)$ is the least common multiple of $2\pi/B$ and $2\pi/C$.

   (*e*) The graph of $y = \sin x + \cos(2\pi x)$ is not periodic. The period of $y = \sin x$ is 2π, and the period of $y = \cos(2\pi x)$ is 1. The least common multiple of two numbers a and b, LCM(a, b), is defined as the smallest positive number for which there exist integers m and n such that $am = bn = \text{LCM}(a, b)$. Thus, there can be no least

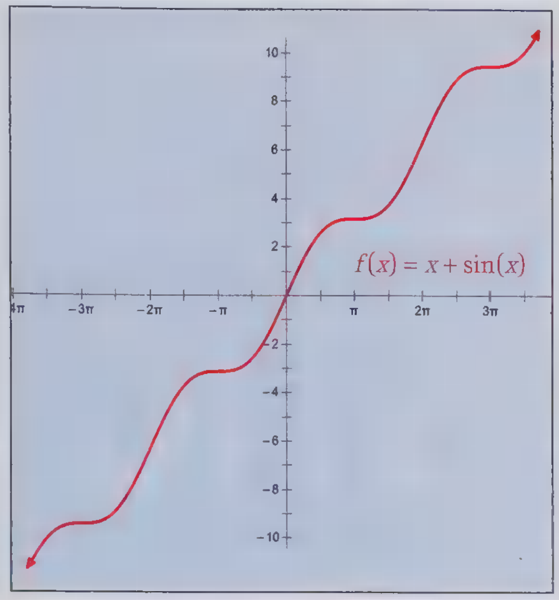

Fig. 4 The function $y = x + \sin x$ is not periodic.

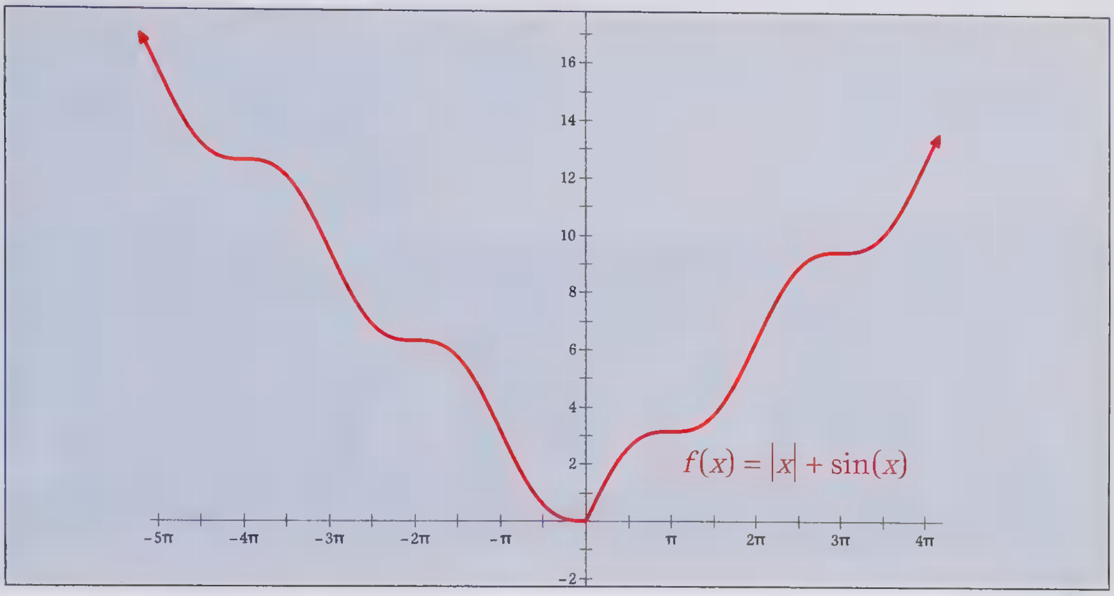

Fig. 5 This function is neither periodic nor symmetric to the y-axis.

common multiple of a rational and an irrational number.

4. (*a*) **Figure 4** shows the graph of $y = x + \sin x$ on the interval $[-3\pi, 3\pi]$. This function is not periodic because the sine curve "climbs up" the line $y = x$. There is no value of p for which $f(x) = f(x + p)$.

   (*b*) **Figure 5** shows the graph of $y = |x| + \sin x$ on the interval $[-5\pi, 4\pi]$. This function can be defined piecewise as $y = -x + \sin x$ for $x < 0$ and $y = x + \sin x$ for $x \geq 0$. The two branches of the function are not symmetric with respect to the y-axis.

   (*c*) **Figure 6** shows the graph of $y = 2^x + \sin x$ on the interval $[-9\pi, 3\pi]$. The range of $y = \sin x$ is $[-1, 1]$. For negative values of x, 2^x is less than 1, and the graph is dominated by the behavior of the sine curve. For positive values of x, 2^x grows rapidly and dominates the graph, which shows little effect of the addition of the sine values.

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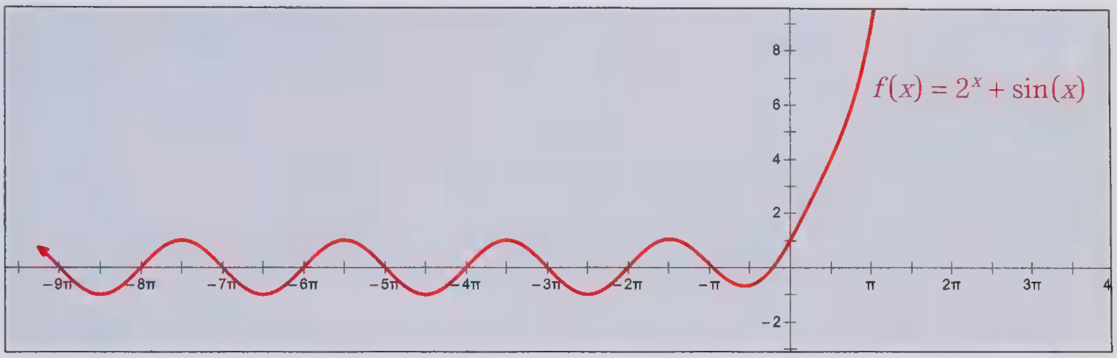
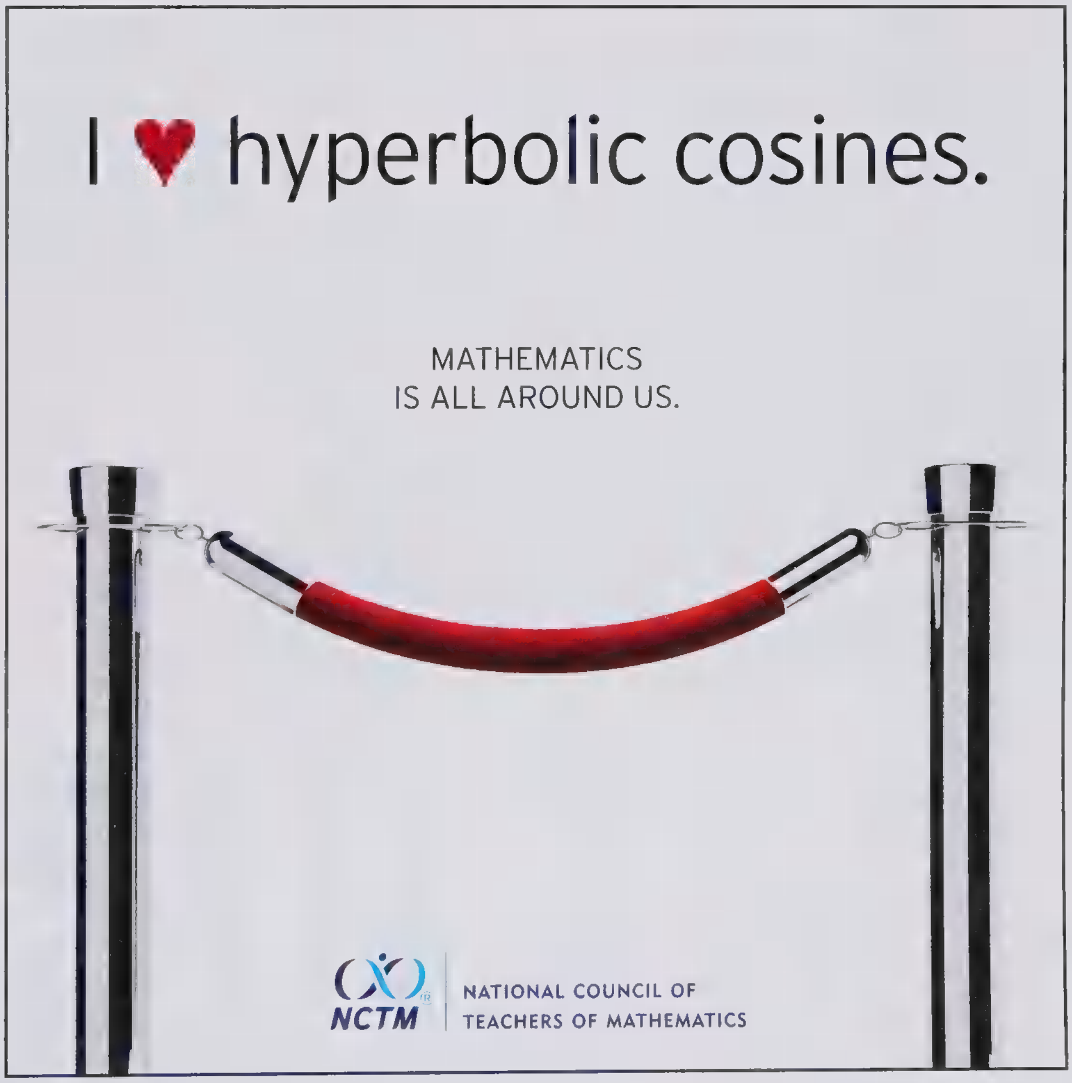


Fig. 6 This function is not periodic.



PREPARE FOR
MORE REALISTIC

The potential drop in reported state proficiency rates when the new CCSSM assessments are implemented will require adjusted expectations.

Matthew R. Larson and Steven Leinwand

Educators in forty-five states and the District of Columbia are hard at work interpreting and implementing the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010). This work typically involves teacher participation in professional development activities focused on developing an understanding of the content standards as well as the standards for mathematical practice. Across the country, educators are also analyzing the model content frameworks, item prototypes, and achievement level descriptors being released by the two national assessment consortia: The Partnership for Assessment of Readiness for College and Careers (PARCC) and Smarter Balanced Assessment Consortium (SBAC). Although mathematics teachers still have to prepare their students for current state assessments, many educators are beginning to ask—with justifiable anxiety, given the consequences attached to student performance—how their students might perform when the new assessments are first administered in the 2014–15 school year.

PREDICTING PARCC AND SBAC RESULTS

Results will ultimately depend on a variety of factors, including how PARCC and SBAC performance standards required for proficiency will be set. However, there is strong evidence that educators nationwide should expect significant reductions in the percentage of students deemed proficient when compared with the proficiency rates currently reported by states using their own assessments.

A case in point is the nagging concern that during the No Child Left Behind era nearly all states set low proficiency standards, as evidenced by the discrepancy between the proficiency percentage reported on the National Assessment of Education Progress (NAEP) and those reported by individual states. For example, in 2009 only Massachusetts had a state standard for proficient performance in grade 8 mathematics equivalent to the NAEP standard (Bandeira de Mello 2011, p. 13) (see **fig. 1**). Every other state's standard for proficient performance on its state assessment is lower than the NAEP standard and results in somewhat to significantly higher reported rates of proficiency.

A report prepared by the American Institutes for Research (AIR) (Phillips 2010) compared the mathematics proficiency standards in each state with the international benchmark used in the Trends in International Mathematics and Science Study (TIMSS). Comparing current state proficiency standards with international benchmarks is instructive because one criterion for the development of the Common Core State Standards for

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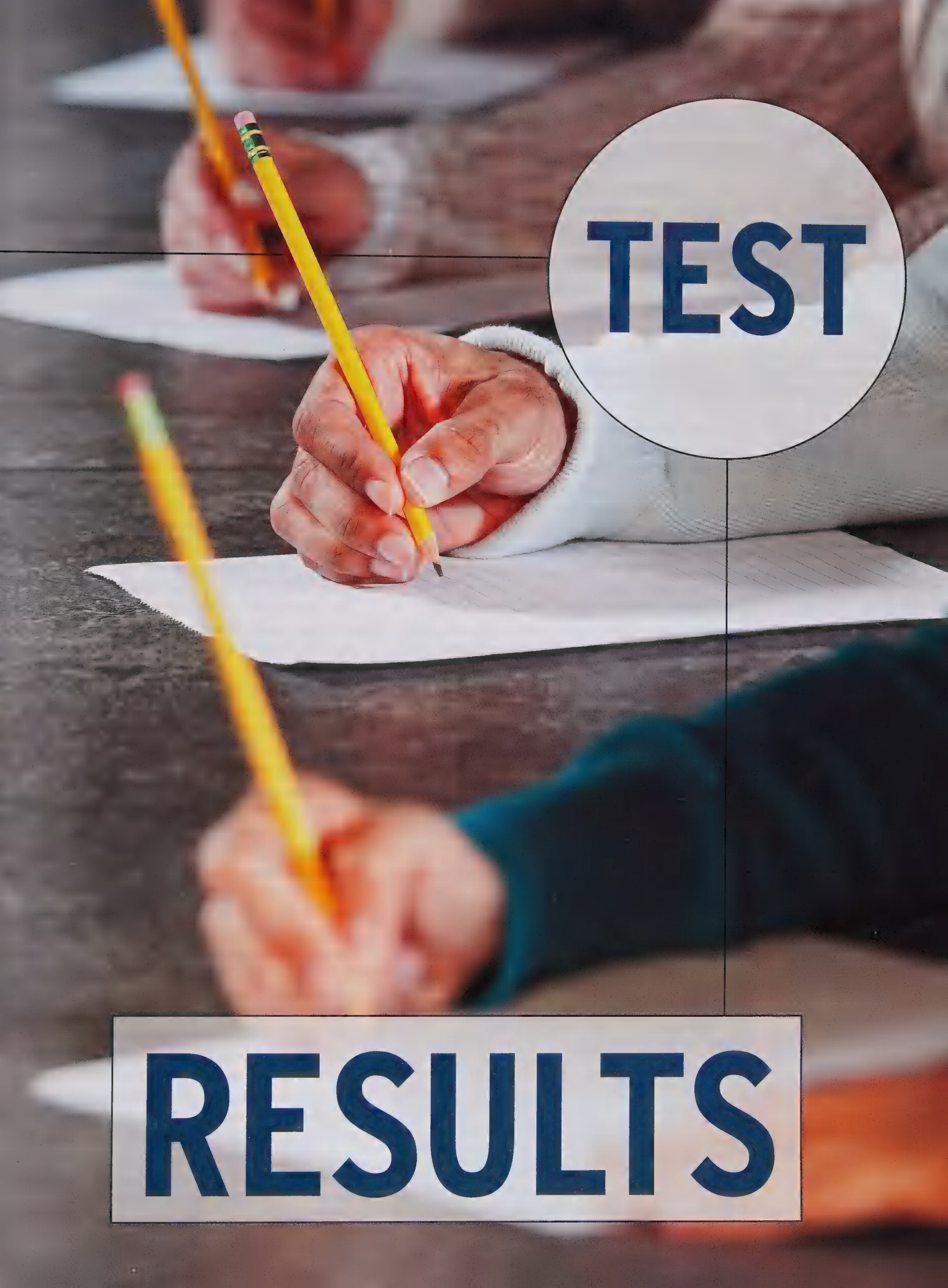
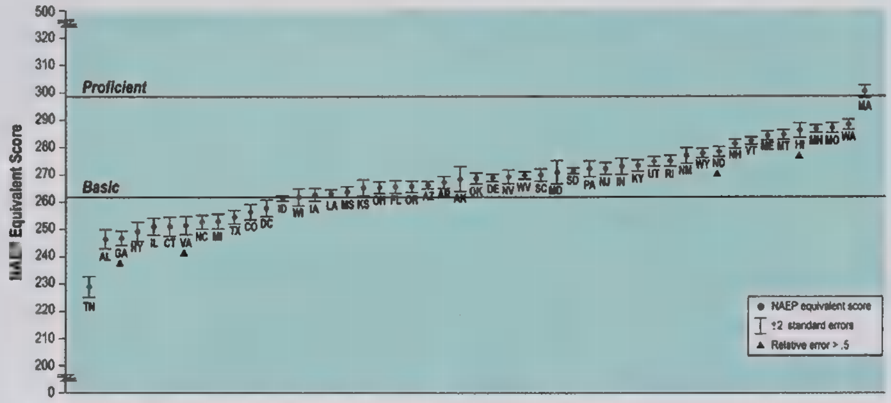


TEST RESULTS



▲ Inferences based on estimates with relative error greater than .5 may require additional evidence.

Source: Bandeira de Mello (2011)

Fig. 1 This graph shows NAEP scale equivalents of state grade 8 mathematics standards for proficient performance by state in 2009.

Mathematics was that the standards be internationally benchmarked.

Eighth-grade mathematics proficiency as reported by states under the requirements of No Child Left Behind in 2007 can be compared with an estimate of percentage proficient if the states had used a high but not advanced internationally benchmarked common standard. This comparison showed that the mean eighth-grade state mathematics proficiency rate would drop from 62 percent to 29 percent and would drop in each of the forty-eight states included in the study with the exception of Massachusetts and South Carolina (Phillips 2010).

During the 2011–12 school year, Kentucky administered its new K-PREP statewide assessment, which was designed to be representative of the Common Core State Standards. At the middle school level in 2010–11, Kentucky reported 65 percent of students proficient in mathematics, but that figure dropped to 40.6 percent proficient under the new assessment in 2011–12 (Ujifusa 2012). This drop is similar to the mean estimated drop in the AIR report and, taken together with the other statistics, may indicate that most states will experience a significant drop in their mathematics proficiency rates when the new assessments are implemented.

PREPARING FOR THE LIKELIHOOD OF LOWER PROFICIENCY RATES

Given these expected and, in some cases, significant drops in the percentage of students deemed proficient, educators at the state, district, and building levels must begin the process of preparing stakeholders for this likelihood to mitigate the panic and overreaction that might occur when results of the new assessments are released for the first time in 2015. Many school and district leaders have neither the time nor the subject-matter expertise to anticipate and appreciate the nuances of these new results. Thus, classroom mathematics teachers need to be aware of and clearly understand the reasons behind the potential drop in proficiency rates and begin to educate their school leaders about this likelihood.

What should the key messages be? The critical message, no matter how difficult it is for many people to accept, is this: Most states have set relatively low performance standards, and current proficiency rates reported under No Child Left Behind do not adequately reflect what students need to know and be able to do in mathematics to compete internationally (Phillips 2010). Exacerbating the problem is the fact that most state assessments under No Child Left Behind have a propensity to assess mathematical skills in isolation at a low-level depth of knowledge (Herman and Linn 2013) and have not assessed mathematical processes in addition to content as outlined in the CCSSM. In other words, current state proficiency rates under No Child Left Behind in many states inflate students' true level of mathematical understanding when measured against an international performance standard that defines mathematical proficiency in terms of connected mathematical understandings and processes in addition to procedural skills. We need to confront this fact and move forward from a new but more realistic baseline of student achievement.

Additional actions and important messages that need to be developed, communicated, and taken include the following:

- Comparisons to past scores on state assessments will have little value. Results of PARCC and SBAC will reflect the performance of a new assessment, with new standards, set to a higher performance standard.
- States and school districts that have adopted teacher evaluation systems tied to student performance on assessments will need to consider that any decrease in the percentage of proficient students as measured by PARCC or SBAC is likely due to a change in the performance standard under the new assessments and not to a decrease in instructional effectiveness.
- Because many states still use paper-and-pencil assessments and the new assessments will be administered through a digital platform, mathematics teachers will need to provide students with experience taking mathematics assessments online to prevent any potential drop in performance due to the change in the assessment platform.
- School boards and the public need to understand that improvement in proficiency rates under the new assessments will take time. Meaningful improvement in teaching and learning is a complex endeavor that will take time and support to achieve.
- Because it will take time to implement the CCSSM and for students to develop the habits

of mind outlined therein, mathematics teachers cannot wait until 2014–15 to begin the process. It can begin immediately, and if teachers work collaboratively to interpret and implement the CCSSM (Kanold and Larson 2012), mathematics teachers can successfully begin the process even before schools or districts begin formal implementation efforts.

- Adopting higher content standards and setting a higher performance standard are essential if we are to give our students the opportunity to learn the mathematics that they need to become productive members of society and to compete in a marketplace that is increasingly global in nature.
- Parents will need to hold their students "to the highest standards that push them out of their comfort zones" (Friedman and Mandelbaum 2011, p. 124). Unless students engage in meaningful mathematical work, both inside and outside school, the goals of higher achievement under the CCSSM will be impossible to achieve. Mathematics teachers can immediately begin to share this message with parents through curriculum nights, parent-teacher conferences, and newsletters.

PERSEVERANCE ISN'T JUST FOR STUDENTS

If the results of the new assessments of the CCSSM result in lower proficiency rates, many knee-jerk reactions may occur. It will be easy for teachers to become discouraged. It will be easy for school administrators to overreact and implement counterproductive practices in an effort to find quick fixes. It will be easy for school board members to remove school leaders to demonstrate that they are taking action. It will be easy for parents to believe that their child's school is failing. It will be easy for business leaders to use lower scores to point to the "failure" of the educational system. And it will be easy for policymakers to declare the CCSSM a failure. None of these reactions is likely to be helpful or to improve the teaching and learning of mathematics.

Standard for Mathematical Practice 1 states that students will "make sense of problems and persevere in solving them" (CCSSI 2010, p. 6). The standards for mathematical practice are processes that students are expected to engage in as they learn the content standards. However, we need to recognize that to achieve the vision of higher mathematics achievement for all students, perseverance will be a critical attribute not only for students but also for the entire system. All those involved in educating students and with an interest in their success will need the perseverance and courage to accept that previous scores were artificially high and work from a new baseline to support better teaching and learning for all students.

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Editor's note: This manuscript was submitted in response to the call "On the Front Burner: Emerging Issues in Mathematics Education."

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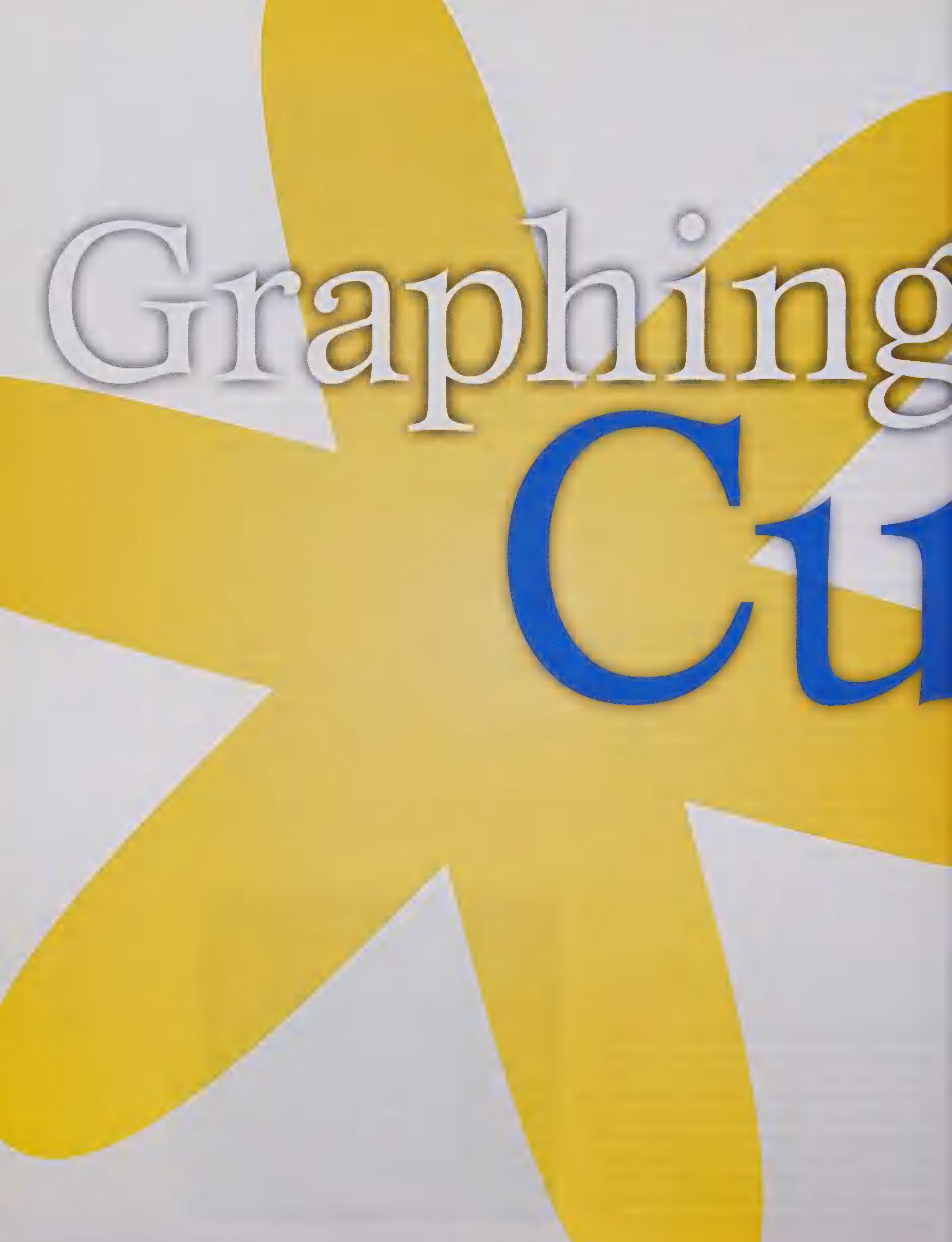
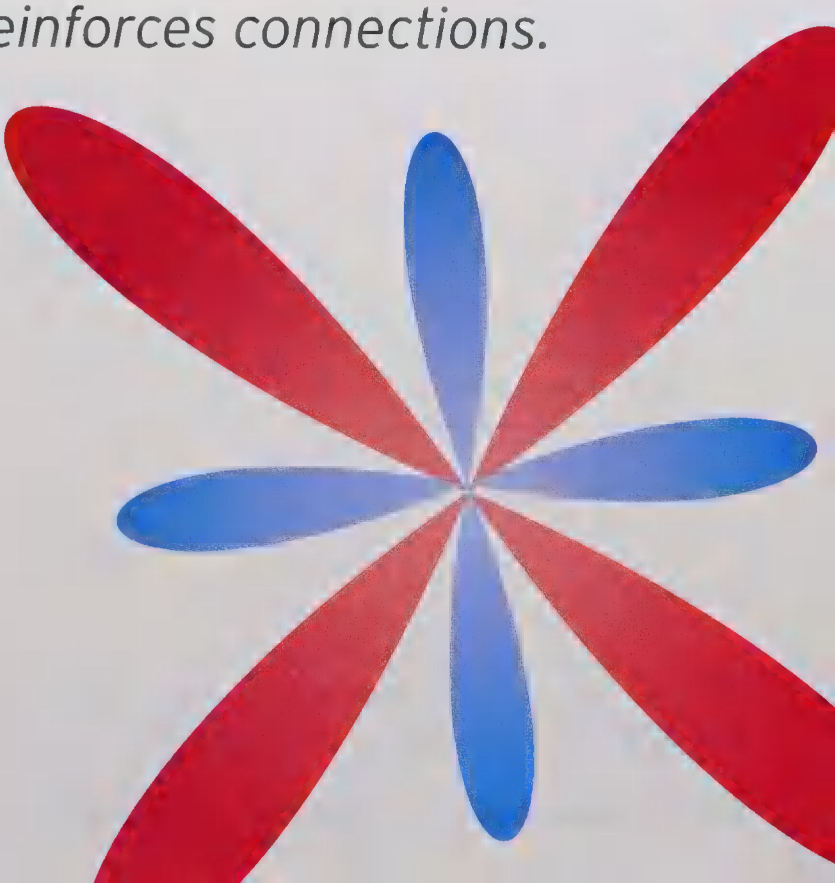


Graphing Cu

Polar rves

Jonathan F. Lawes

Graphing the polar function on a rectangular plane simplifies graphing, increases student understanding, and reinforces connections.



If you teach mathematics long enough, you will probably get the opportunity to show students how to graph polar curves. This can be a daunting task. Some aspects of the polar coordinate system are easy for students to master, whereas others are very difficult. Graphing calculators can quickly produce magical-looking polar curves. However, if students do not have a solid understanding of the underlying principles, the curves on the calculator display are just that—magic.

Graphing polar curves typically involves a combination of three traditional techniques, all of which can be time-consuming and tedious. However, an alternative method makes producing a polar curve relatively simple, increases students' understanding of the polar coordinate system, and reinforces graphing techniques learned earlier in trigonometry.

Before attempting to graph polar curves, students should be taught the fundamentals of the polar plane—for example, the vocabulary, the clockwise measurement of positive angles, how to plot points. Students should see enough demonstration and should practice enough so that they are proficient at plotting any polar point, including positive and negative values for both the radius, r, and the angle, θ. Learning to plot points is a crucial step in helping students understand the polar coordinate system. Once ample time has been devoted to helping students develop this essential skill, they are prepared to learn how to graph polar curves. We will look first at traditional techniques and then introduce an alternative method that both incorporates and improves on these.

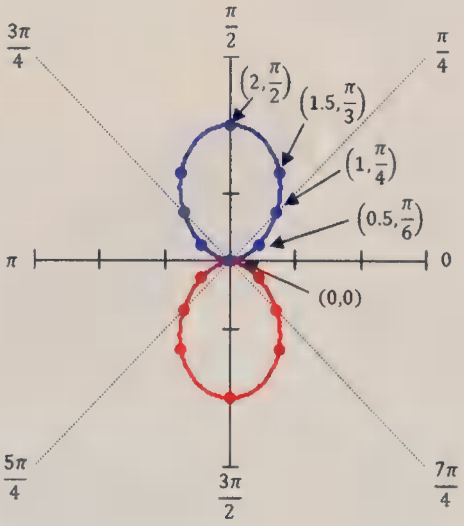
TRADITIONAL TECHNIQUES: TABLE OF VALUES, SYMMETRY, AND MEMORIZATION

Making a table of values is a good initial approach when learning to graph any new function. The same is true when learning to graph polar curves. Students begin by making a table of values for θ and r and then selecting values of θ to substitute into the polar function. The function is then evaluated to find the radius for each angle. Common angles (such as $\pi/6$, $\pi/4$, $\pi/2$, and their multiples) are often used. When the table is complete, students plot the points on the polar plane as (r, θ) pairs and connect them in sequence to create the polar curve. We will use the polar function $r = 1 - \cos(2\theta)$ to demonstrate this technique (see **fig. 1**).

Depending on the complexity of the function, students may need to choose as few as eight or as many as several dozen values of θ to produce an accurate graph. Students may also find that using only common angles does not provide a complete picture. They may need to fill in the gaps in their table (and their graph) by adding less common angles, such as $\pi/12$, $\pi/8$, and so on. In addition, students must not only choose appropriate values of θ but also correctly evaluate the function for each angle they choose. A single calculation error can severely distort the graph and lead to frustration. Producing a curve using this technique can also take a great deal of time, causing some students to become discouraged and to fail to see the value of the polar coordinate system. Graphing polar curves using this technique is often more an exercise in perseverance than a discovery. However, students should see at least two or three examples of graphs created by using a table of values.

θ	$r = 1 - \cos(2\theta)$
0	0
$\pi/6$	0.5
$\pi/4$	1
$\pi/3$	1.5
$\pi/2$	2
$2\pi/3$	1.5
$3\pi/4$	1
$5\pi/6$	0.5
π	0
$7\pi/6$	0.5
$5\pi/4$	1
$4\pi/3$	1.5
$3\pi/2$	2
$5\pi/3$	1.5
$7\pi/4$	1
$11\pi/6$	0.5
2π	0

(a)



(b)

Fig. 1 The color-coded points in the table (**a**) have been graphed, producing the polar curve $r = 1 - \cos(2\theta)$ (**b**).

After graphing a few polar curves using a table of values, students typically look for an easier technique. A calculator could be used to create the table, but students still need to plot the points. Moreover, if they use their calculators to produce the points, many will simply push a few more buttons to produce the graph. This approach robs them of a valuable opportunity to make their own connections and discoveries. Using a calculator is probably more appropriate once students have a fuller understanding of how polar curves are formed. Of course, the teacher can always decide at what point and to what degree a calculator should be used to improve and speed any exploration or calculation.

Two additional techniques can reduce the number of points that need to be plotted to produce a complete graph. One technique is to check for one of three types of symmetry on the polar plane. By checking for symmetry with the polar axis, the line $\theta = \pi/2$, and the pole itself, students can plot a small number of points and then use symmetry to produce the remainder of the curve. Checking for symmetry can be a difficult process. For example, one test for symmetry with the line $\theta = \pi/2$ is to replace θ with $\pi - \theta$ in the polar equation and then simplify. If an equivalent expression results, the curve is symmetric with the line $\theta = \pi/2$.

Let's look at a typical example that might be used to demonstrate the usefulness of symmetry. When graphing the cardioid $r = 1 - \sin\theta$, we first check for symmetry. We notice that if we replace θ with $\pi - \theta$, we get the following:

$$\begin{aligned} r = 1 - \sin(\pi - \theta) &= 1 - [\sin\pi\cos\theta - \cos\pi\sin\theta] \\ &= 1 - [0 \cdot \cos\theta - (-1) \cdot \sin\theta] \\ &= 1 - \sin\theta \end{aligned}$$

Because replacing θ with $\pi - \theta$ in $r = 1 - \sin\theta$ yields an equivalent expression, the curve is symmetric with the line $\theta = \pi/2$. Thus, we can construct a table of values for angles in quadrants I and IV, plot them on the polar plane, and then use symmetry to produce the rest of the curve (see **fig. 2**).

Finding symmetry reduces the number of points needed to plot a complete curve and also provides a good application for trigonometric identities. However, it is a challenging task, especially if students check for all three types of symmetry. In addition, asking students to understand symmetry of polar curves before they have a great deal of experience using them is a stretch for most students.

The final traditional technique is to memorize the general form of the equations that produce the standard polar curves that appear in most textbooks: circles, roses, limaçons, cardioids, and lemniscates. This approach involves memorizing how different trigonometric functions, coefficients, operations, and arguments produce various polar curves so that they can be recognized and graphed quickly. For example, students might memorize how to determine the number of petals a rose will have or the difference between a limaçon and a cardioid.

Consider the following descriptions. Roses are typically produced by equations of the form $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$ where $a, n \neq 0$. If n is even, the rose has $2n$ petals. If n is odd and not equal to ± 1, the rose has n petals. In contrast, equations such as $r = a - b\cos\theta$ produce a limaçon with an inner loop when $0 < a < b$. However, if $0 < b < a$, the same equation produces a limaçon with no inner loop. Committing to memory the characteristics of the standard polar equations and the curves they produce can be overwhelming.

The traditional techniques of using a table of values, symmetry, and memorization can be valuable tools when working with polar curves. However, the arduous nature of these techniques can deter even the best students from developing a solid understanding of polar curves. Many students fail to make the connection between the characteristics of the polar equation (amplitude, period, maximum radius, zeros, etc.) and the polar curve produced by that equation. The techniques are often just more memorization, tedious calculations, and unconnected skills. Having students use calculators for exploration, prediction, and discovery of polar curves can be fun, quick, and useful, but their fundamental understanding of the polar graphing process might still be left incomplete.

AN ALTERNATIVE METHOD: GRAPHING THE POLAR FUNCTION ON A RECTANGULAR PLANE

An alternative method produces the correct graph in a simple and effective manner while improving students' fundamental understanding of how the polar coordinate system works. Further, this

θ	$r = 1 - \sin\theta$
$-\pi/2$	2
$-\pi/3$	≈ 1.87
$-\pi/6$	1.5
0	1
$\pi/6$	0.5
$\pi/2$	2

(a)

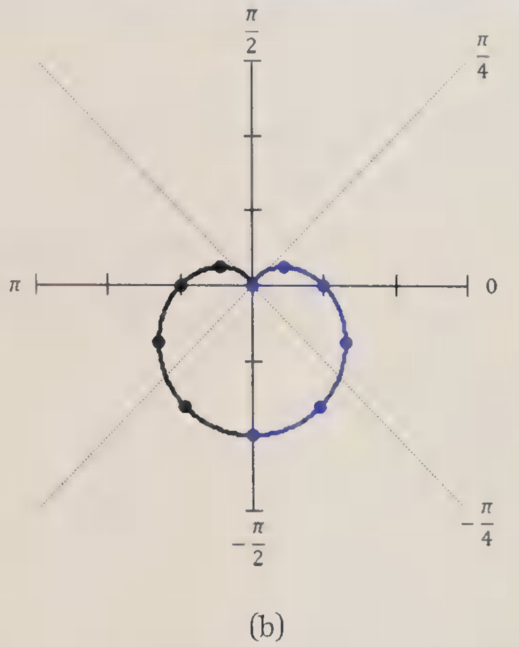


Fig. 2 The blue points on the polar curve $r = 1 - \sin\theta$ are taken from the table (**a**); the black points are graphed by symmetry (**b**).

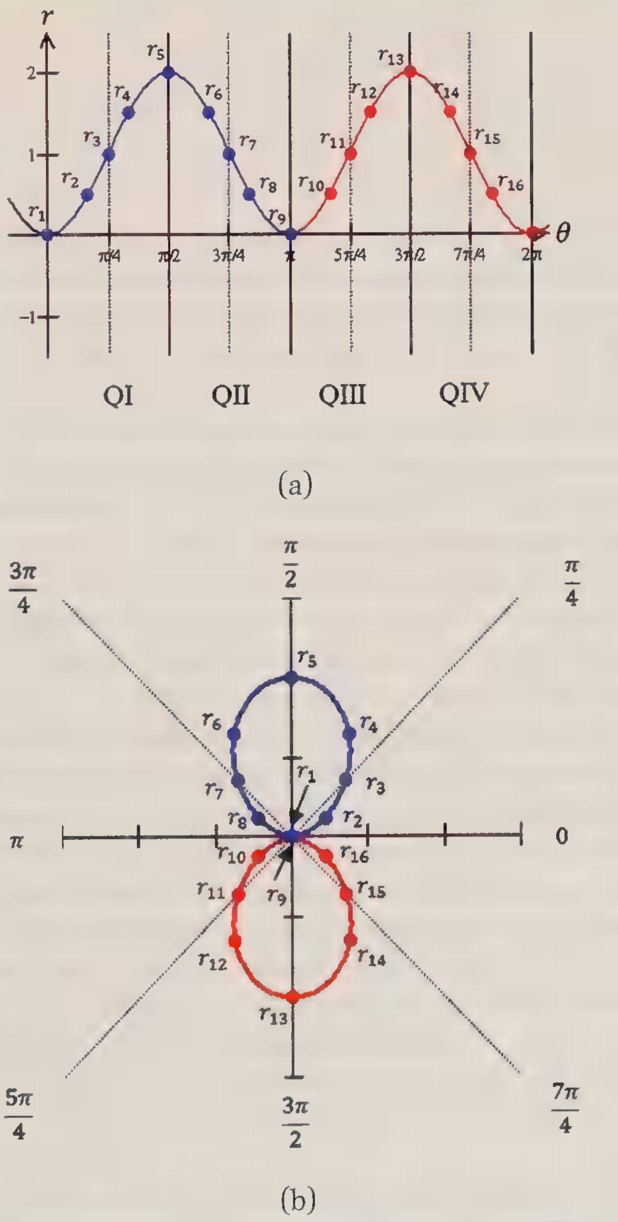


Fig. 3 The rectangular graph for $r = 1 - \cos(2\theta)$ has been sketched and color-coded, and sixteen radii have been identified (**a**), producing the polar graph shown in (**b**).

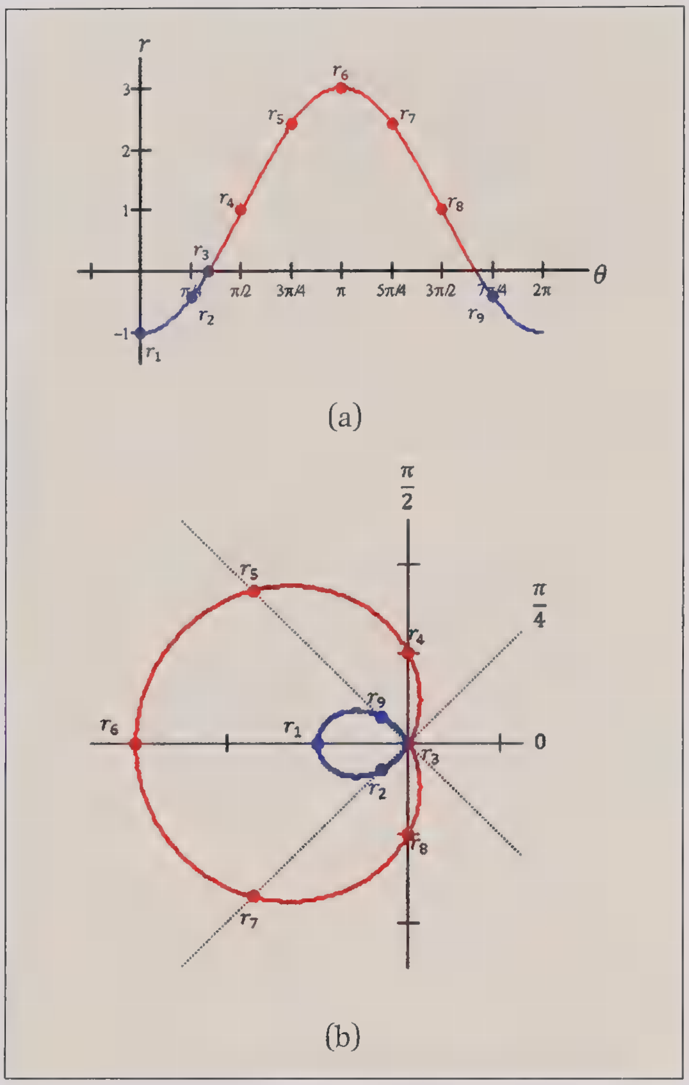


Fig. 4 The positive and negative radii for $r = 1 - 2\cos\theta$ have been color-coded, and nine points have been identified (**a**); these have been plotted in the polar plane (**b**).

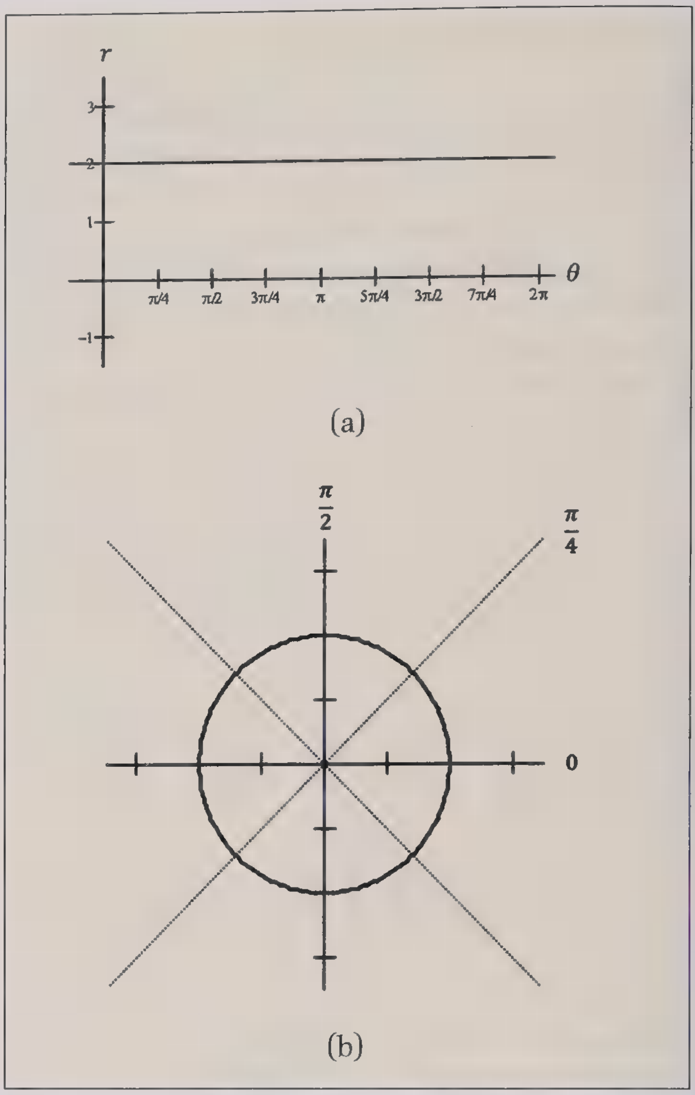


Fig. 5 The rectangular graph for $r = 2$ clearly shows that the radius is always 2, regardless of the value of θ (**a**); transferred to the polar plane, the graph is a circle (**b**).

method reinforces principles learned previously when graphing transformations of trigonometric functions. It involves graphing the polar function on a rectangular plane before graphing it on the polar plane (Essary 1982).

Graphing the polar function in more familiar rectangular coordinates gives students the ability to quickly sketch nearly any polar functions that they might encounter, even those that would take a prohibitive amount of time if done by hand using traditional techniques. This method also allows students to visually determine critical characteristics of the polar curve and make connections to previously learned graphing skills. Instead of producing a table of exact values, the alternative method uses rectangular graphing to produce a "picture" of *all* the values contained in such a table. We will start with the first example graphed earlier.

To graph $r = 1 - \cos(2\theta)$ on the polar plane, we first graph $r = 1 - \cos(2\theta)$ on a rectangular plane. We label the horizontal axis θ and the vertical axis r. The values on the horizontal axis represent the angle to which students turn from standard position to plot the point; the values on the vertical axis represent the radius for the corresponding value of θ. This approach gives students a direct connection to the skills they learned when graphing transformations of trigonometric functions. The (θ, r) points on the rectangular graph can then be plotted as (r, θ) points on the polar plane to produce the polar curve (see **fig. 3**).

By examining the behavior of the graph in **figure 3a** on the interval from 0 to π radians (the blue portion of the graph), students can see that the radius increases from a minimum of 0 at $\theta = 0$ to a maximum of 2 at $\theta = \pi/2$ and then decreases to 0 at $\theta = \pi$. When graphing the polar curve in **figure 3b**, students can see that the blue petal is created as θ increases from 0 to π radians and that the radius changes from 0 to 2 and then back to 0. The formation of the red petal can be seen in a similar manner. After completing several graphs using this method, students find that they can graph each petal by identifying the zeros and the maximum radius of the function on the rectangular graph.

Students can also look for similarities between quadrants. For example, in **figure 3a** the portion of the graph in quadrants III and IV behaves similarly to the portion of the graph in quadrants I and II, respectively. Once students graph the polar curve

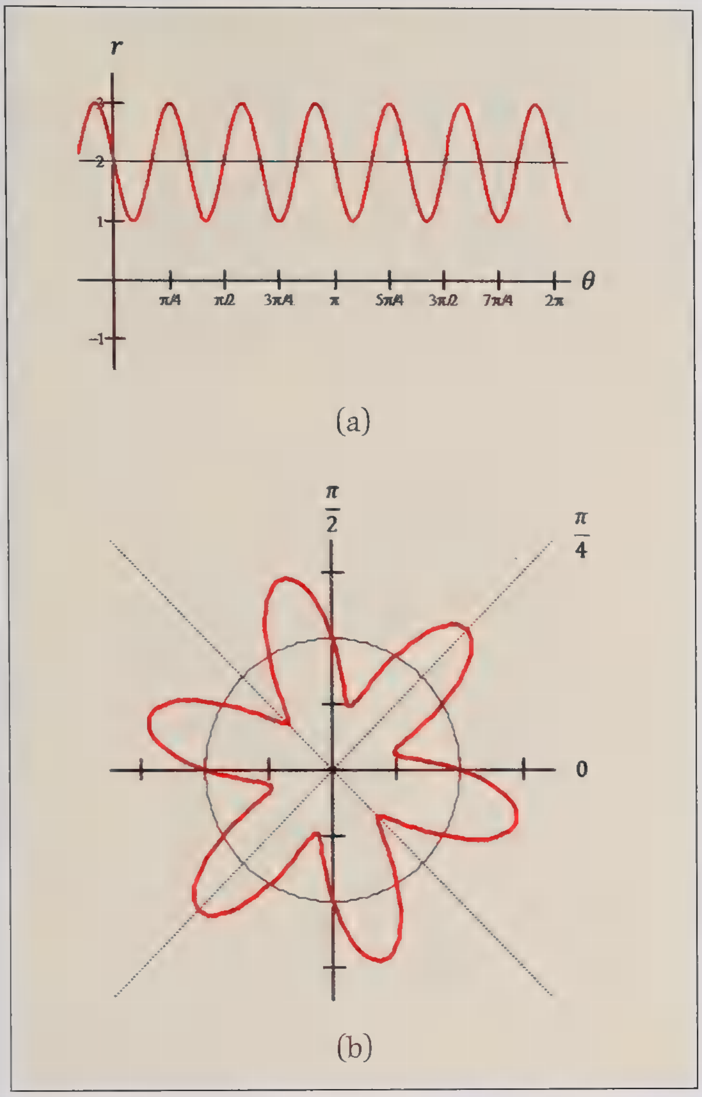


Fig. 6 The rectangular graph for $r = 2 - \sin(6\theta)$ varies above and below $r = 2$ (**a**), producing the polar curve shown in (**b**).

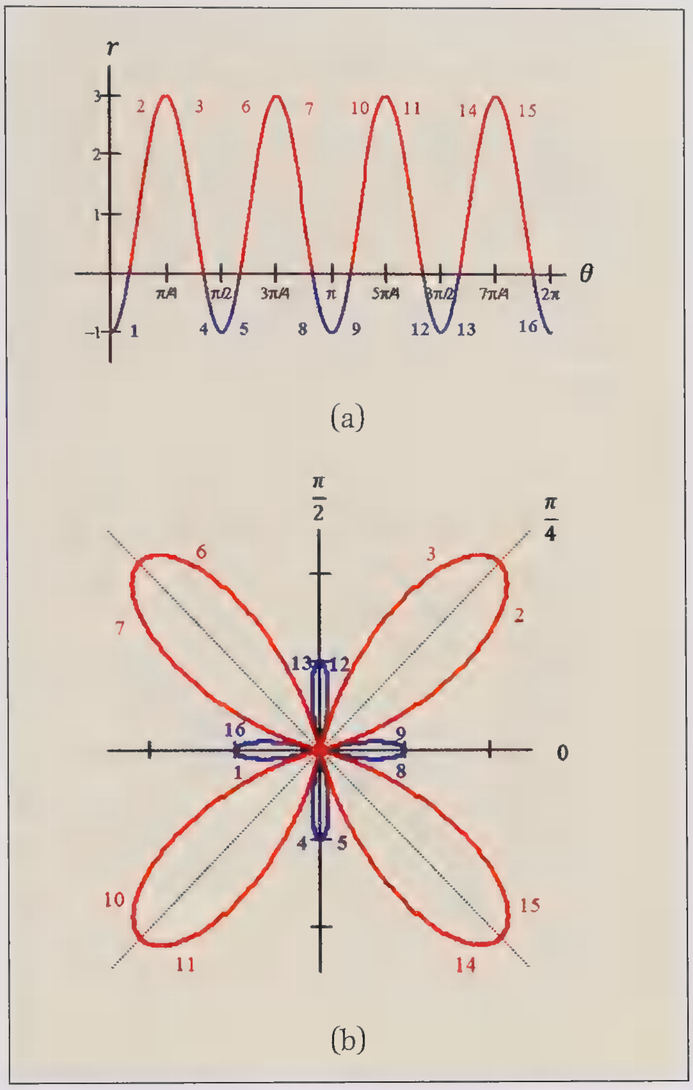


Fig. 7 The magnitude, sign, and numbering of the color-coded portions of the rectangular graph of $r = 1 - 2\cos(4\theta)$ (**a**) helps obtain the polar graph (**b**).

in quadrants I and II, they can quickly produce a similar shape in quadrants III and IV. “Seeing” these symmetries is often easier than checking for symmetry algebraically and can help students produce a more accurate polar curve.

All the radii in the rectangular graph shown in **figure 3** are above the θ-axis—that is, the radii are positive—so they are graphed in the direction of the terminal side of θ, measured from standard position. However, if the radii are below the θ-axis—that is, if the radii are negative—then they are graphed in the direction opposite the terminal side of θ (i.e., in the direction of $\theta + \pi$).

The next example develops a better understanding of how more difficult polar curves are created. It also demonstrates the importance of the zeros of the rectangular function and the sign of the radius.

Consider the polar function $r = 1 - 2\cos\theta$. As an equation, this example looks similar to the first example: $r = 1 - \cos(2\theta)$. Looks, however, are deceiving; the curves, both rectangular and polar, are fundamentally different.

In **figure 4a**, students can see that the radii below the θ-axis are negative and smaller than the radii above the θ-axis. These smaller radii are what create the small inner “loop” (the blue portion) shown in **figure 4b**. The rectangular graph gives students the ability to quickly graph a reasonable sketch of the polar curve without getting caught up in the details of finding an exact value of θ or r at all but the most important points. Whether the zero radius, r_3 (see **fig. 4a**), occurs at $\pi/3$ (the exact value) or at $3\pi/8$ (a convenient estimate) is not critical to the overall shape of the polar graph (see **fig. 4b**). Finding such exact values can always be done when needed, rather than on every problem.

This method of graphing has another advantage. When students have a quick, accurate way of sketching polar curves, they enjoy exploring how both standard and nonstandard polar curves are created. Consider the following examples. Students can see that for $r = 2$ in **figure 5a**, regardless of the angle, the radius is always 2, so a circle of radius 2 centered at the pole is formed in **figure 5b**. By combining the simple function in **figure 5a** with $\sin(6\theta)$ to obtain $r = 2 - \sin(6\theta)$, students can quickly produce the graph in **figure 6a**. They can see that the radius begins with a value of 2 at $\theta = 0$ but then varies from 1 to 3 in a sinusoidal fashion. This variation creates the polar curve in **figure 6b**

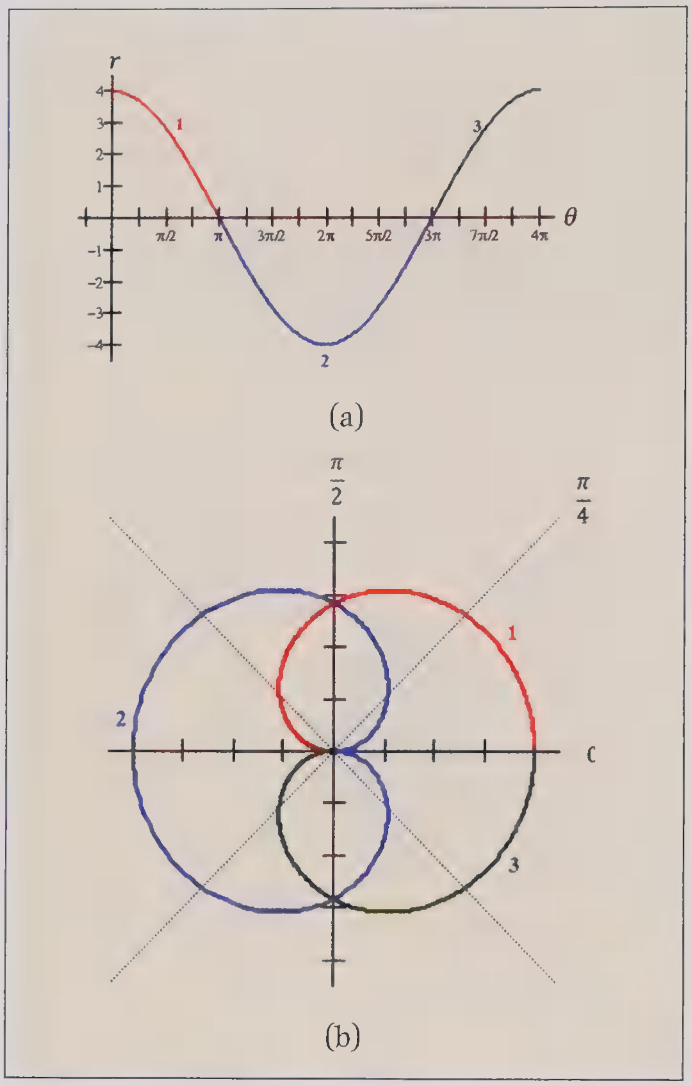


Fig. 8 Students combine the color-coded portions of the rectangular curve $r = 4\cos(\theta/2)$ (**a**) with those of the polar curve (**b**).

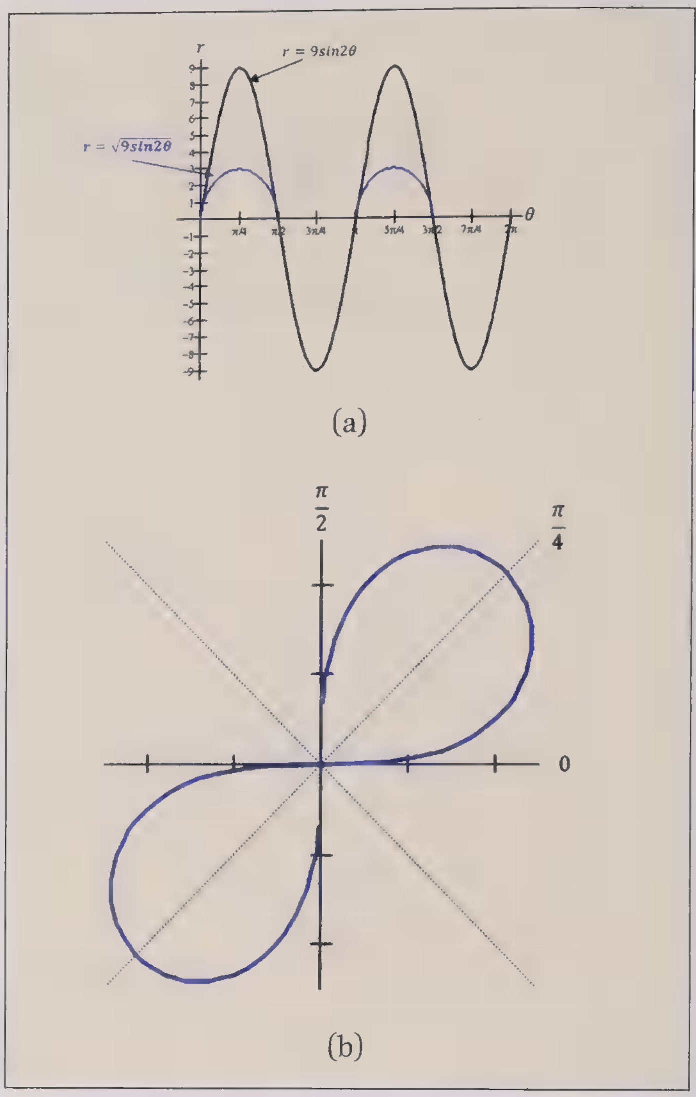


Fig. 9 The rectangular graph of $r = 9\sin(2\theta)$ helps students quickly sketch the graph of $r = \sqrt{9\sin(2\theta)}$ (**a**); the blue portions are graphed to produce the polar curve $r^2 = 9\sin(2\theta)$ (**b**).

and can help students understand how changes in the radius influence the shape of the polar curve.

Using a rectangular curve with some points below the θ-axis, as in the next example, students can see how the magnitude and sign of the radius determine why the curve is graphed where it is. Consider the equation $r = 1 - 2\cos(4\theta)$. The corresponding numbered segments have been added to **figure 7** to help identify important features.

Recently, after just three examples had been presented to teach this method to a trigonometry class, students suggested that we try graphing a curve I had never considered: $r = 4\cos(\theta/2)$. The students easily identified the amplitude (4 units) and the period (4π units) of the function and quickly produced the graph in **figure 8a**. With little previous practice using this method, most students were able to produce the graph in **figure 8b**. Some students stopped the polar graph at 2π but were eventually able to see why their graph was not complete.

Lemniscates, which are of the form $r^2 = a^2\sin(2\theta)$ or $r^2 = a^2\cos(2\theta)$, can be problematic for students. Determining how and why these equations make the shapes they do can be difficult to understand even with the aid of a graphing calculator. Using a rectangular graph of the polar equation can help make the behavior and intricacies of lemniscates more clear.

For instance, to produce the polar curve for the relation $r^2 = 9\sin(2\theta)$, we begin by removing the square and graphing the function $r = 9\sin(2\theta)$, shown as the black graph in **figure 9a**. Taking the approximate square roots of the function values of $r = 9\sin(2\theta)$ produces the blue graph ($r = +\sqrt{9\sin(2\theta)}$) (see **fig. 9a**). The coordinates of the points on the blue graph can then be transferred onto the polar plane to produce a complete polar curve (see **fig. 9b**). Note that because the function values of $r = 9\sin(2\theta)$ are negative on the intervals $(\pi/2, \pi)$ and $(3\pi/2, 2\pi)$, taking the square root would yield nonreal solutions. Therefore, no points are graphed on the polar plane for these values of θ (i.e., θ-values in quadrants II and IV).

On further analysis, we realize that because the polar equation $r^2 = 9\sin(2\theta)$ is a relation, we should also account for negative solutions when taking the square roots of the function $r = 9\sin(2\theta)$. Doing so produces the red graph ($r = -\sqrt{9\sin(2\theta)}$) shown in **figure 10a**. For a given value of θ, the radii for the blue and the red graphs in **figure 10a** differ only

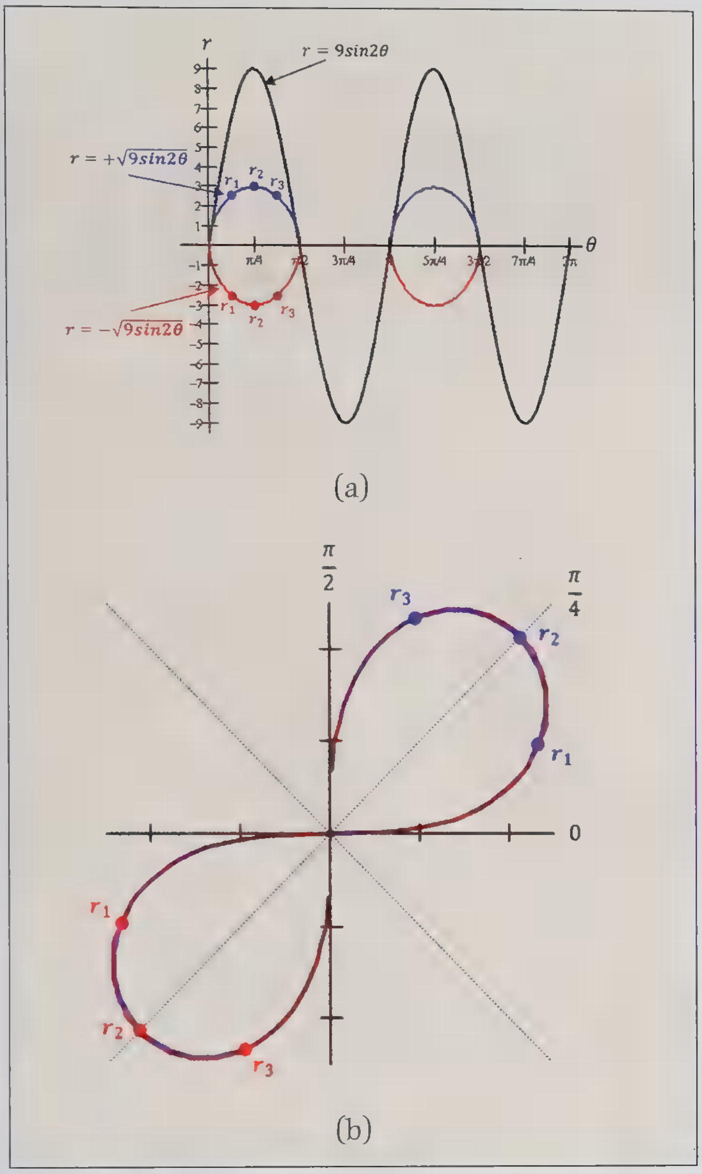


Fig. 10 The rectangular graphs of $r = 9\sin(2\theta)$ and $r = \pm\sqrt{9\sin(2\theta)}$ (**a**) can be sketched quickly and then regraphed in polar coordinates (**b**). Because the blue and red graphs in (**a**) differ only in sign, the curve $r^2 = 9\sin(2\theta)$ is traced twice on $[0, 2\pi]$.

by their sign; thus, transferring the blue and the red coordinates between the angles of 0 and $\pi/2$ produces the entire polar curve. That is, between 0 and $\pi/2$, the positive blue radii produce the petal in quadrant I, whereas the negative red radii produce the petal in the opposite direction (quadrant III), seen in **figure 10b**.

Continuing to plot coordinates through 2π will show that each petal is traced twice, once in red and once in blue. Because of this retracing, plotting both positive and negative radii for each angle is not required to produce this particular polar curve. However, depending on the relation and the period, both positive and negative radii may be necessary to produce a complete graph. Although not a lemniscate, another relation, $r^2 = 9\sin(3\theta)$, provides a good opportunity for exploring the complex behavior of this type of polar curve.

Numerous characteristics of polar curves can be observed and learned using this rectangular method. In addition, advanced questions can be explored. For instance, does using angles from 0 to 2π always produce a complete polar curve? If not, what interval will? How should the θ-axis be scaled to make plotting the polar curve easier and more accurate? What are the maximum and minimum radii, and at what angles do they occur? How would a phase shift affect the polar curve? Calculus students could explore the question, Where do two polar curves intersect? Once students understand polar curves, they enjoy applying their knowledge to explore the answers to such questions.

The traditional techniques of plotting points, symmetry, and memorization are useful tools; they are a vital part of understanding polar curves. However, graphing the polar function on the more familiar rectangular plane removes much of the drudgery of the other techniques and gives students the ability to produce accurate polar curves quickly and easily. It also builds on other graphing techniques that they have previously learned and increases their understanding of the polar coordinate system. This increased understanding allows students to explore and discover fundamental properties of the polar system that may be too difficult otherwise.

REFERENCE

Essary, Alice. 1982. "Rectangular Aids for Polar Graphs." *The Two-Year College Mathematics Journal* 13 (3): 200–203.

Editor's note: For more on polar graphing, see "Exploring Polar Curves with GeoGebra," by Tuyetdong Phan-Yamada and Walter M. Yamada III (Technology Tips, *MT* Oct. 2012, vol. 106, no. 3, pp. 228–33).

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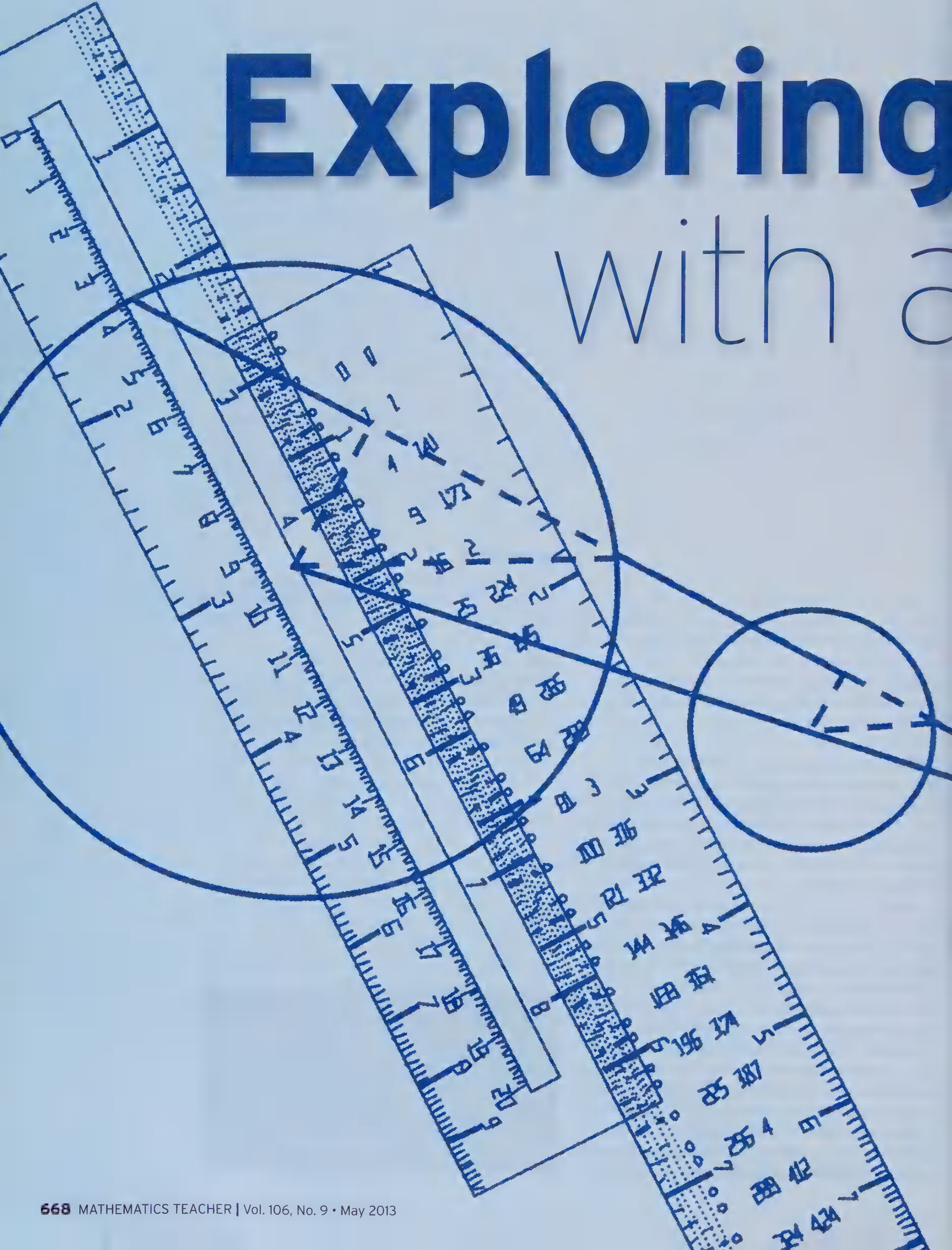
Exploring with a



Logarithms Ruler

Processes using linear measurement can be adapted to teach complex topics such as polynomial multiplication, rational exponents, and logarithms.

Elliott Ostler

Once upon a time every advanced algebra or calculus textbook included tables of logarithms. This was a time when instant access to the numbers they contained was important for computations related to exponents and exponential functions and there was no other efficient way to complete these calculations. Since then, digital technologies have made doing logarithmic calculations by hand obsolete. Although computing logarithmic values with a calculator represents an appropriate use of technology, the removal of tables or logarithms from textbooks has meant the loss of an important mathematical context.

Logarithms are still important to learn but now for a different reason. The need to multiply numbers using logs has certainly diminished, but other applications of exponential and logarithmic functions remain important, particularly those relating to topics such as radioactive decay, bacterial growth, acidity levels in chemical solutions, and even population dynamics. Given that logarithms are still a critical part of the secondary school curriculum, students should know how they were originally generated and used for computation. They should be able to see how logarithm tables were organized and used and that the tables were derived from ideas as basic as linear measurements.

I am not suggesting that students should go back to a less efficient method of computation, but the ability to examine log values in tabular form helps develop and refine information about the incremental and predictable nature of logarithms. Patterns such as linear progressions in vertical columns and product-sum relationships can be discovered in the logarithm table values, and investigating those patterns helps provide context for the broader topic of logarithms. Seeing the numbers organized in tabular form allows us to consider a bigger picture and is important inasmuch as a graph is important in creating information about the y-values of a function at specific values of x. Without these numeric and graphical contexts, students are often left wondering whether the values provided by the calculator are correct.

With a humble admission that we are traveling down a well-researched road, we will assume that there are still some interesting things to discover along the way and take a step back in time to consider logarithms within a context that everyone knows: the ruler. The impulse to measure things is a natural one, and we practice the concept early in our lives, but have students ever considered that logarithm values can result from physical measurements? Of course, this possibility makes sense, given that the tables were originally developed without the use of modern technology. Specifically, this article demonstrates two interesting applications of how logarithms relate to measurement: (1) as a method for multiplying whole numbers using number lines; and (2) as a way to calculate rational exponents by measurement.

The applications of logarithms can be confounding to students because logs appear in a somewhat mysterious and less familiar kind of notation. As mathematics instruction slowly transitions from the "how" to the "why," we are able to observe students' discomfort with logarithm-based questions that require more than procedural fluency, such as those relating to compound interest on a bank account or calculating the intensity of an earthquake. From a typical chapter on logarithms, students are expected to apply the rules of logs, expand or condense algebraic expressions involving logarithms, and solve equations involving logarithms. Students are also often instructed that logarithms represent exponents and that the logarithm notation denotes the inverse of an exponential function. But even if students can conquer mechanical fluency, many still are confused about what all the procedures really mean and how the logarithm concepts can be recognized and applied in more advanced mathematics or in other disciplines such as physics, biology, or chemistry.

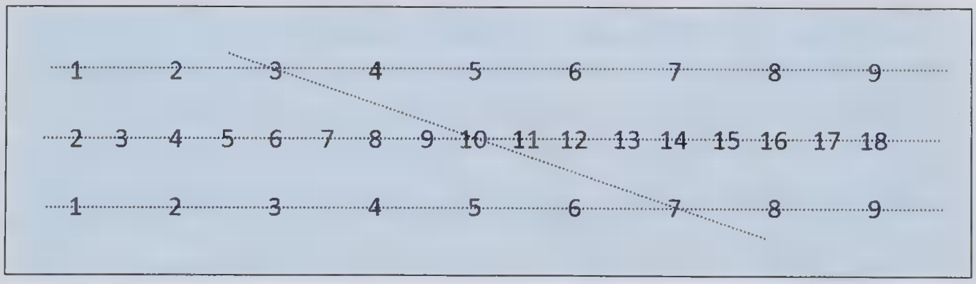


Fig. 1 The addition 3 + 7 = 10 is illustrated with this nomograph.

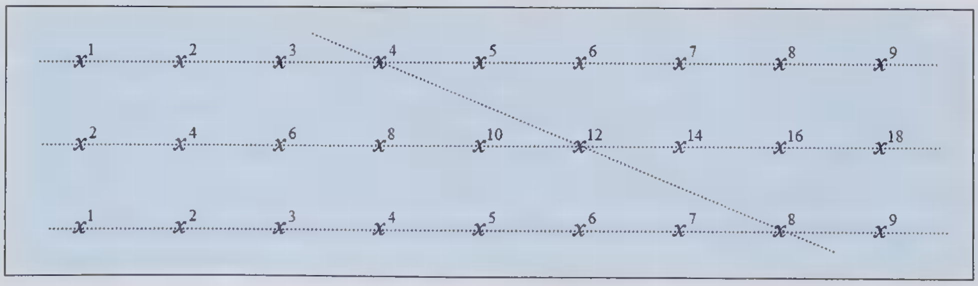

x1 x2 x3 x4 x5 x6 x7 x8 x9
x2 x4 x6 x8 x10 x12 x14 x16 x18
x1 x2 x3 x4 x5 x6 x7 x8 x9

Fig. 2 This example illustrates $x^4 \cdot x^8 = x^{12}$.

ADDITION WITH A SET OF NUMBER LINES

If we begin with a basic and familiar form of geometric addition using an addition nomograph, a logical transition to the rules of exponents and logarithms emerges. A *nomograph* is simply a set of number lines that work side by side to illustrate a geometric method of adding. Considering the conceptual diagram representing the three number lines in **figure 1**, note how a transversal passing through any two addends on the outside number lines crosses the middle line at the sum.

Let's examine how this nomograph works. Because the middle line (where we find the sum) is halfway between the lines where the two addends are located, the distance along the middle line to the point where the result is found is essentially the average of the distances along the outer lines. The average of 3 and 7 is (3 + 7)/ 2, or 5. Because the sum of 3 and 7 is twice the average of these numbers, the points on the middle line then need to have numeric values twice as large as those on the outer lines to show an accurate sum.

Students generally think that a nomograph is a "cool" way of seeing addition geometrically, especially because their early experience with geometric addition typically consists of using a single number line and adding or subtracting by counting to the right or left. Students who often still struggle with fraction operations will ask whether the same procedure can be used with addition and subtraction of fractions. With a little exploration, students discover (usually without being told) that they can count by halves or fourths on each number line instead of using integer units, thus making the same process work for fractions. Of course, there is nothing magical in this computation or even in how it is represented, but watch how the concept evolves.

GRAPHING EXPONENTS AS DISTANCES

A follow-up question after students discover the fraction nomograph is, Can a similar nomograph technique be used for multiplication? The answer becomes clear as students realize that the multiplication process is essentially based on addition. Students should recall from earlier algebra that $x^n \cdot x^k = x^{n+k}$ where the variables n and k represent real-number exponents. This algebraic property provides a process for multiplying algebraic quantities with the same base by adding exponents, so let's look at how that process is developed. We will use number lines similar to those that worked for addition in our first example, but the distances along the lines now correspond to the exponents of the numbers they represent (this concept typically takes a while for students to understand, so it helps to have a ruler at the ready as the example shown in **fig. 2** is demonstrated). Consequently, if number lines are expressed as increasing powers of some base, the same addition property shown in **figure 1** should hold true.

Now, if we assign a numeric value greater than 1 to the variable x (the value for x must be greater than 1 since 1 raised to any power will never increase and values smaller than 1 will shrink), a multiplication nomograph is the result. The substitution of actual numeric values for the variables on each number line drives home the point for students. For example, we can use any value for the base, but for now, to keep the numbers as small as possible, we will let $x = 2$ and then evaluate each resulting exponent along the number line (see **fig. 3**).

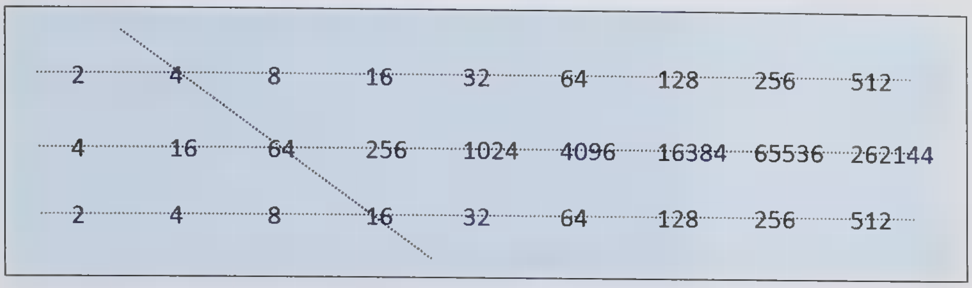


Fig. 3 This multiplication nomograph illustrates 4 • 16 = 64.

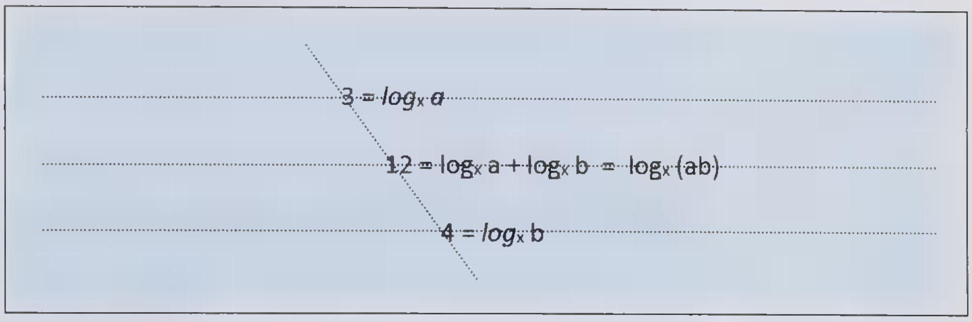


Fig. 4 The measurements in this nomograph are represented using logarithm notation.

TRANSITIONING FROM EXPONENT DISTANCES TO LOGARITHMIC DISTANCES

Ultimately, the exponent addition rule serves as an introduction to the properties of logarithms. In the example illustrated in **figure 4**, we have supposed that some measured distance a on a number line corresponds to x^3 and that another distance b on the opposite number line corresponds to x^4. If $a = x^3$, then $\log_x a = 3$; similarly, $b = x^4$ and $\log_x b = 4$. The measurements of a and b on the nomograph are now represented using logarithm notation, and the product of a and b can be found by adding their logs.

It is inconvenient to limit the base to 2, as we did in **figure 3**, especially because we do not typically use 2 raised to increasing powers when we do calculations. What if we wanted to multiply 7 by 13? Students often find it difficult to identify exact locations for 7 and 13 on the top and bottom

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The *MTMS* Word Problem

The Editorial Panel of *Mathematics Teaching in the Middle School* is asking readers to help solve the *MTMS* Word Problem:

Which word or words should be used in a new journal title?

Just as *Mathematics Teacher* and *Teaching Children Mathematics* are descriptive without using the labels "high school" or "elementary school," *MTMS* might better serve a wider range of educators without the words "middle school." What better source for a new title than our readers? Get creative, and send us your favorite names. The rationale of the Editorial Panel and more information are available online at **www.nctm.org/mtmswordproblem**.

Submit your most creative names through **www.nctm.org/mtmswordproblem** by **May 15, 2013.** The Editorial Panel will select four to six choices that will be voted on by *MTMS* readers in September 2013. The most popular name will be presented to the NCTM Board of Directors. Those who submitted the four to six nominated names will each win a $50 gift certificate to the NCTM bookstore. The individual who submits the winning entry will receive an iPad® and will be featured in the first issue of the newly named journal in August 2014.

number lines and even more difficult to interpret an accurate result on the middle line using 2 as a base. This difficulty, however, is the impetus to create something more accurate, so additional exploration is necessary.

CREATING A TABLE OF LOGARITHMS FOR A NONSLIDING SLIDE RULE

The logical assumption for improving our $x = 2$ exponent nomograph would be to use 10 as a base and follow the procedure illustrated in **figure 4**. To do this, let's examine an abbreviated, approximated log table (see **fig. 5**) and use these values to calculate the appropriate distances for each number on a number line (or a standard ruler). These values can be used to turn a grid line or a ruler of any length into a logarithm ruler. Using graph paper helps students put the numbers in the correct positions and also reinforces the transition to decimal measurements. Students should note that all the calculated values are approximate measurements and that we are approximating to four decimal places when we construct our makeshift log table. In doing so, we can pretend that the number lines are rulers.

$\log 1 = 0$	The number 1 would be at the left end of the ruler.
$\log 2 = .3010$	The number 2 would be .3010 of the way down the ruler.
$\log 3 = .4771$	The number 3 would be .4771 of the way down the ruler.
$\log 4 = .6021$	The number 4 would be .6021 of the way down the ruler.
$\log 5 = .6990$	The number 5 would be .6990 of the way down the ruler.
$\log 6 = .7782$	The number 6 would be .7782 of the way down the ruler.
$\log 7 = .8451$	The number 7 would be .8451 of the way down the ruler.
$\log 8 = .9031$	The number 8 would be .9031 of the way down the ruler.
$\log 9 = .9542$	The number 9 would be .9542 of the way down the ruler.
$\log 10 = 1.00$	The number 10 would be on the right end of the ruler.

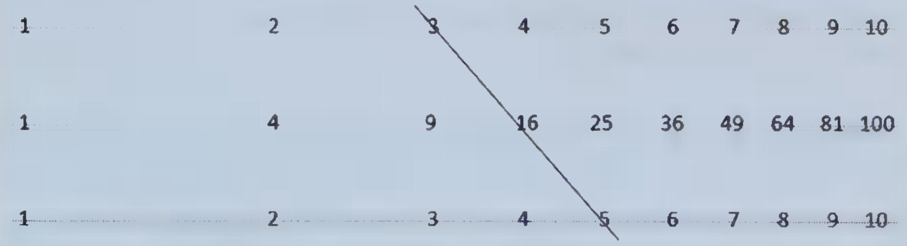

Fig. 5 The number lines serve a dual purpose.

Suppose: $5^x = 8$ then $\log_5 8 = x$
Change of Base: $\log 8/\log 5 = x$

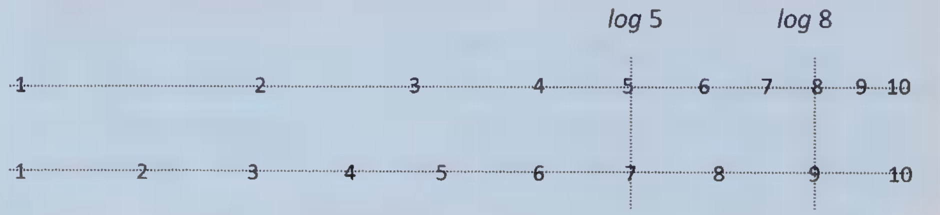

Fig. 6 Dividing 9 by 7 produces a very good approximate value for x in $5^x = 8$.

The marks on the middle number line can be derived by applying the rules of logarithms (assume now that we are using a common log with a base of 10). Numbers on the exterior lines can be added to obtain the quantity on the center line, so $\log n + \log n = 2 \log n = \log n^2$. We have essentially created part of a good old-fashioned slide rule, but it does not need to slide to work. In **figure 5**, where $3 \cdot 5$ is represented, the transversal crosses at a point just shy of 16 on the middle number line, which we would have to interpret as 15. Larger numbers can be multiplied as well if the log table is completed and marked with greater precision.

For example, if we wanted to multiply 23 by 37, we would need the graphical positions of $\log 2.3$ and $\log 3.7$ on the outside nomograph lines; these distances would be read as two-digit numbers. The center line, where the product would be located, would also need to be completed with equal precision and interpreted as the appropriate three-digit product. Numbers of any size can be multiplied in this fashion. However, as the numbers become larger, locating the position on the center number line that represents the exact product also becomes more difficult because the numbers and the marks that represent the products are so close together.

CALCULATING FRACTIONAL EXPONENTS WITH A RULER

If we line up one of the log scales from **figure 5** with a linear scale (which could even be a standard 12-inch ruler), we can calculate rational exponents by measuring and dividing the logarithmic distances using a geometric interpretation of the change-of-base formula. Because $2^3 = 8$, then $\log_2 8 = 3$, which we can verify as $\log 8/\log 2 = 3$. Recall that $\log 8$ and $\log 2$ are both measured distances that can be divided to give us the exponent of 3: $\log 8 \approx 0.903$; $\log 2 \approx 0.301$; and $0.903/0.301 = 3$.

Let's try a more difficult problem by lining up a log ruler and a linear ruler and solving the equation $5^x = 8$. We know that $5^1 = 5$ and that $5^2 = 25$, so the unknown exponent should fall somewhere between 1 and 2 (and should probably be closer to 1).

By measuring the log scale using the bottom number line, we can see that $\log 5$ lines up with a number on the linear scale that represents a distance of about 7 and that $\log 8$ lines up with a number close to 9 (see **fig. 6**). If we calculate 9/7 as the resulting exponent, we get approximately 1.2857, and if we use a calculator to compute $\log 8/\log 5$, we get approximately 1.292—not a bad estimate. The estimate gets better if we are more accurate with our linear measurement. The scale of our linear ruler does not matter because the ratio of distances of $\log 8$ to $\log 5$ is always the same, regardless of the linear scale being used.

More complex examples—such as dividing numbers and extracting roots—can be solved with additional exploration, but the idea of using measurement to understand logarithms is pure enough to spark some discussion about how logarithms might apply elsewhere. With such technology applications as The Geometer's Sketchpad®, we can use different linear scales to calculate fractional exponents. We can even explore logarithmic calculations using newer technologies such as an app for the IPad®.

MEASUREMENT IS FUNDAMENTAL

Measurement is one of the most fundamental ideas taught in early mathematics. Thus, the prospect of exploring complex ideas such as logarithms through measurement is essential to applying mathematics. Students are typically familiar with and very comfortable using tools as basic as a ruler, even if the broader topic of measurement may be more difficult for them. The fact that measurement is such a basic, concrete skill is part of the reason for exploring abstract mathematical ideas such as logarithms with a ruler. If nothing else, students have a mental anchor to remind themselves, "Hey, a logarithm can just be a number on a ruler. I can do that!"

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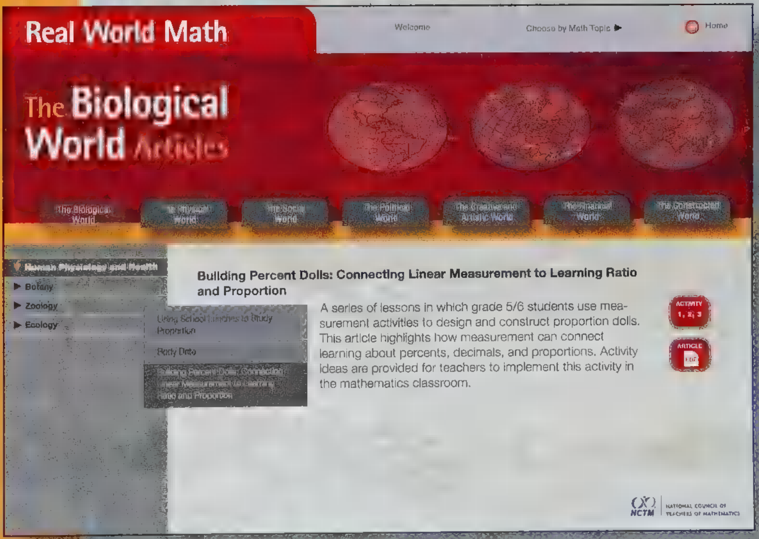


Building Percent Dolls: Connecting Linear Measurement to Learning Ratio and Proportion

A series of lessons in which grade 5/6 students use measurement activities to design and construct proportion dolls. This article highlights how measurement can connect learning about percents, decimals, and proportions. Activity ideas are provided for teachers to implement this activity in the mathematics classroom.

COUNTER

AS STARTING POINTS FOR REASONING

Asked to "fix" a false conjecture, students combine their reasoning and observations about absolute value inequalities, signed numbers, and distance to write true mathematical statements.

David A. Yopp

How do we generate a culture of reasoning and sense making in our classrooms? One way is to encourage students to investigate mathematical statements even after they have found them to be false. Demonstrating that a mathematical generalization is false requires only one counterexample, but sense making demands additional steps.

Such practice is well grounded in the work of mathematicians, whose quest for sense making is continuous. It is not unusual for a mathematician to write a *mathematical proposition* (a statement that he or she believes to be true), only to discover that it is false. A mathematician might then ask whether the proposition can be altered to create a mathematical truth. Doing so while salvaging as much of the original conjecture as possible often involves informal observations and reasoning—one way in which mathematicians discover new mathematics.

Given the call for reasoning and proving to play an important role in school mathematics (NCTM 2000, 2009), as well as growing interest in school mathematics materials that better align with the parent field (RAND 2003), mathematics teachers need articles that document efforts to create classroom experiences that address these lofty goals. This article describes a recent classroom activity with college sophomores in a methods-of-proof course in which students reasoned about absolute value inequalities. The course was designed to meet the needs of both mathematics majors and secondary school mathematics teaching majors early in their college studies.

Addressed in particular is the challenge of navigating the interplay between formal reasoning (e.g., proving) and informal reasoning (e.g., observations from examples) to generate a culture of sense making. NCTM's *Focus in High School Mathematics: Reasoning and Sense Making* (2009) tells us that "mathematical reasoning can take many forms, ranging from informal exploration and justification to formal deduction, as well as inductive observations" (p. 4) and that "reasoning and sense making are intertwined across the continuum from informal observations to formal deductions" (p. 4). This article offers another dimension to this perspective by demonstrating how students can move from formal to informal reasoning and back again and how all types of reasoning can work harmoniously and nonlinearly toward an ultimate objective of making sense of the mathematics being studied.

The topic and mathematics content are appropriate for high school mathematics courses (see Ellis and Bryson [2011] for a discussion of similar content in high school courses). Further, the instructional strategy of asking students to reason about

EXAMPLES AND SENSE MAKING

the examples they provide is relevant for any high school or college mathematics course.

THE TASK

During a unit on inequalities, students were asked to explore the following conjecture:

$$|x - y| \le |x| - |y|$$

The term *conjecture* was used in this course to describe mathematical sentences without any connotation of truth values. The term *proposition* was used to describe statements that someone asserts to be true, and *theorem* was reserved for statements proven to be true. This terminology allowed me to pose questions about mathematical statements to students without giving any hint of whether or not the statements were true.

The quantifiers were purposely left off the conjecture because they are frequently omitted in mathematics textbooks. The students were accustomed to identifying when statements are open statements (such as equations to be solved) and when statements are quantified with an implicit "for all," asserting that something is always true. This particular conjecture was the latter, a for-all statement that can be rewritten more formally as follows:

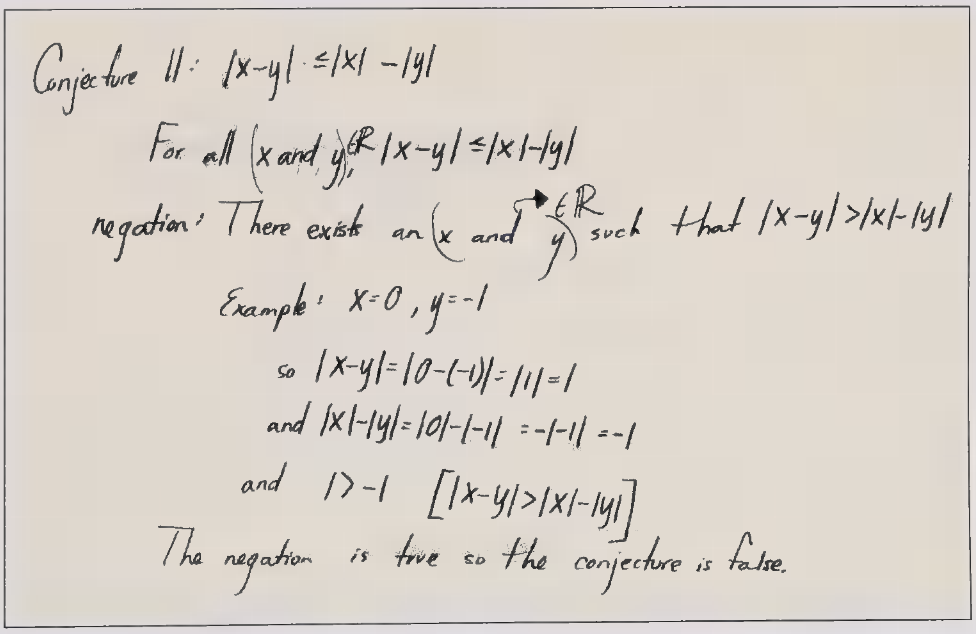
> For all real numbers x and y, $|x - y| \le |x| - |y|$. Equivalently, if x and y are real numbers, then $|x - y| \le |x| - |y|$.

Working individually, students quickly found counterexamples and recorded them. **Figures 1** and **2** are two examples of student work.

Sam correctly asserts that the conjecture is false and provides a suitable counterexample (see **fig. 1**). She also carefully identifies the important formal aspects in her work:

1. She rewrites the conjecture using the appropriate quantifier in the form "for all x, y."
2. She points out that proving the negation of the conjecture (which she writes correctly as a "there exists" statement) demonstrates that the conjecture is false.

Pat also gives an excellent response (see **fig. 2**). He writes the conjecture in proper notation (using the symbol ∀ to denote "for all"), and he writes his counterexample in proper form, noting that his



Conjecture 11: $|x-y| \le |x| - |y|$

For all $(x \text{ and } y) \in \mathbb{R}$, $|x-y| \le |x|-|y|$

negation: There exist an $(x \text{ and } y) \in \mathbb{R}$ such that $|x-y| > |x|-|y|$

Example: $x=0$, $y=-1$

so $|x-y| = |0-(-1)| = |1| = 1$

and $|x|-|y| = |0|-|-1| = -|-1| = -1$

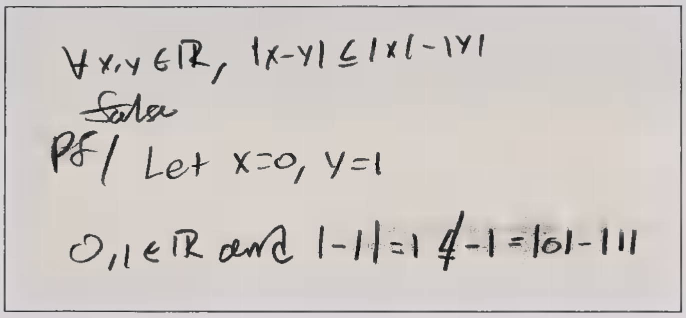
and $1 > -1$ $\left[|x-y| > |x|-|y|\right]$

The negation is true so the conjecture is false.

Fig. 1 If the negation of the statement is true, the statement is false.

choices of values for x and y satisfy the hypothesis, not the conclusion.

Although I was pleased with the students' work, I felt that we could learn more. Even though we had answered the question of whether or not this conjecture is true, as mathematical thinkers we could go further. *Making sense* can be described as understanding the situation, context, or concept by connecting it with existing knowledge (NCTM 2009). Here the context was comparing inequalities that connect to concepts of distance and signed numbers. Conjecturing about how to "fix" the original statement would result in making sense of the original conjecture. By "fix," I meant adjusting the hypothesis or the conclusion so that a theorem (a true proposition) is formed while salvaging as much of the original conjecture's intent as possible.



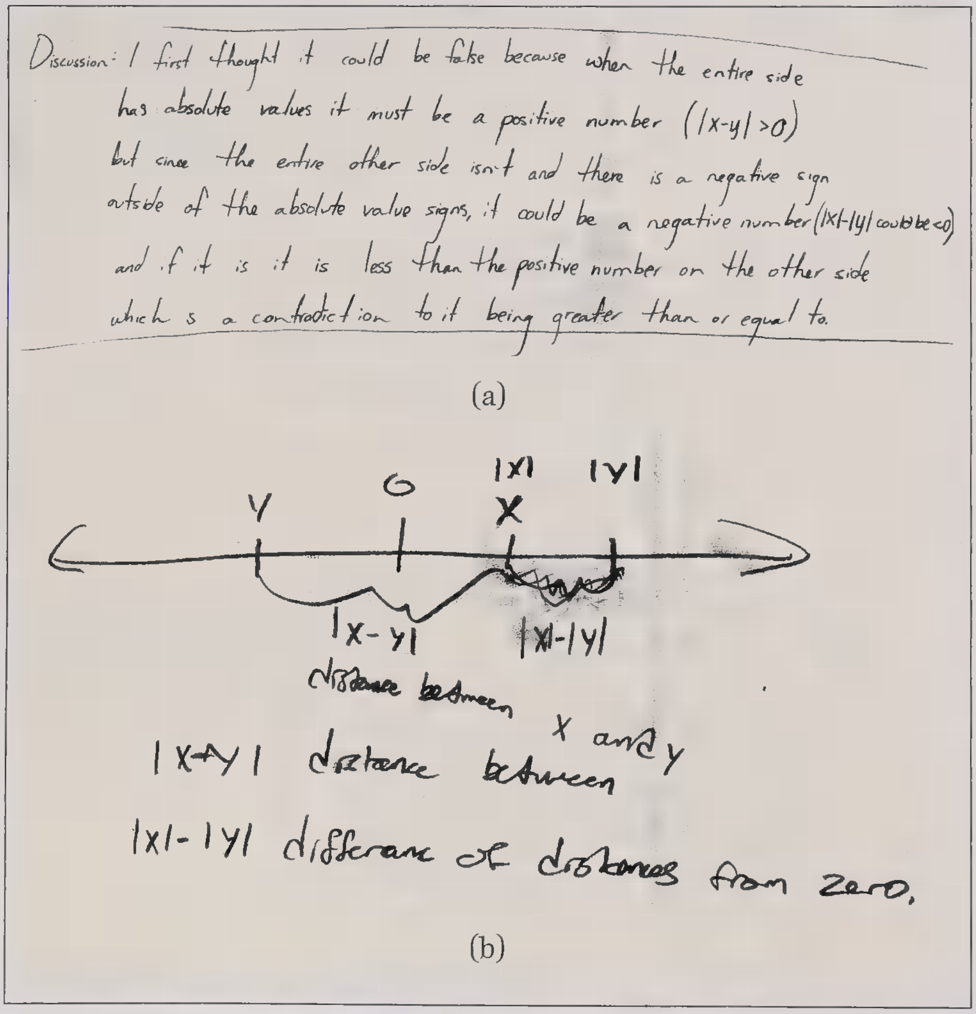
$\forall x,y \in \mathbb{R},\ |x-y| \le |x|-|y|$

False

Pf/ Let $x=0,\ y=1$

$0,1 \in \mathbb{R}$ and $|-1| = 1 \nleq -1 = |0|-|1|$

Fig. 2 Producing a counterexample also shows that the statement is false.



Discussion: I first thought it could be false because when the entire side has absolute values it must be a positive number ($|x-y|>0$) but since the entire other side isn't and there is a negative sign outside of the absolute value signs, it could be a negative number ($|x|-|y|$ could be <0) and if it is it is less than the positive number on the other side which is a contradiction to it being greater than or equal to.

(a)

y
0
|x|
x
|y|
|x−y|
|x|−|y|
distance between
|x−y| distance between x and y
|x|−|y| differance of distances from zero.

(b)

Fig. 3 Student responses tended to fall into the two categories, as the examples here indicate.

This process is obviously subjective because "the original intent" is in the eye of the beholder.

I could have simply asked students to write down a "fixed" statement, but I also wanted them to understand that informal thinking is important in mathematics. Moreover, I wanted them to connect their fix to existing knowledge, which now included observations and counterexamples. In particular, I wanted them to note that although empirical evidence will not prove the modified conjecture (assuming that they wrote a general statement), empirical evidence (i.e., the counterexamples) could be valuable in conjecturing about possible fixes.

I gave students the following directions:

1. Take a moment to reflect on how you found your counterexample. For example, was it a guess-and-check situation, or did you make some sort of observation that led you to a counterexample?
2. Think about what your counterexample tells you about why this statement is true. Summarize your thoughts in a short paragraph.

THE NEXT STEPS

Over the next two evenings, I sorted the students' responses. Student observations that were clearly expressed fell into two categories; a few other responses expressed ideas that were difficult to classify. One type, shown in **figure 3a**, mentioned that one side of the inequality could be negative whereas the other is always positive. Ten responses fit into this category. The other type of clearly articulated response demonstrated that the distance between x and y could be greater than the distance between $|x|$ and $|y|$ (see **fig. 3b**). Three responses fit into this category.

The following is an example of the third type of response:

> The problem is you can provide a counterexample where you can add two positive numbers on one side but subtract one from the other on the other side. And the sum of two positive numbers will always be greater than subtracting one from the other.

This response expresses a reasonable idea, but the approach is a bit misleading, inaccurate, or incomplete because y would need to be negative for $x - y$ to be equivalent to the sum of two positive numbers. The student might have noted this condition but did not explicitly mention it. I placed seven students' responses into this stack of unclear, incomplete, or inaccurate observations. (In hindsight, I wish that I had asked each student in this category to clarify his or her response. I may have overlooked some important thinking.)

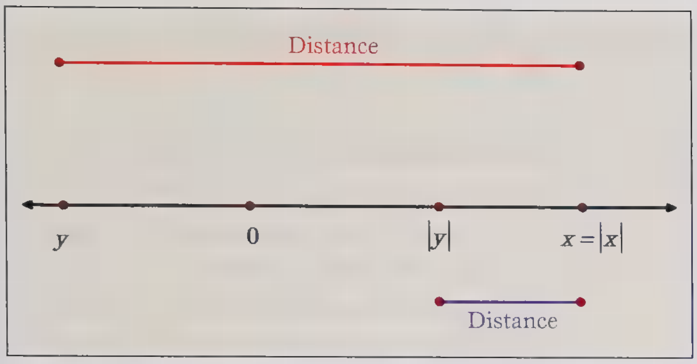


Fig. 4 The two distances can be different if $|x| > |y|$.

Although the approaches represented in **figure 3** are related, these two types of responses are different enough that they could lead to entirely different fixes. This possibility intrigued me, so during the next class period I asked students to develop a fix based on observations from the previous task.

To set the stage, I presented the two responses shown in **figure 3** to the rest of the class and facilitated a discussion about the two types of reasoning. Because Pat's response positions x and y in a manner such that $|x| - |y|$ is negative (see **fig. 3b**), I offered students my graph (see **fig. 4**) to illustrate that the distances Pat discusses could be different even when $|x| > |y|$.

I summarized the discussion by pointing out the following:

1. The right side of the inequality could be negative, which could happen even if x and y are both positive.
2. The distance between x and y could be greater than the distance between $|x|$ and $|y|$ even when both distances are positive.

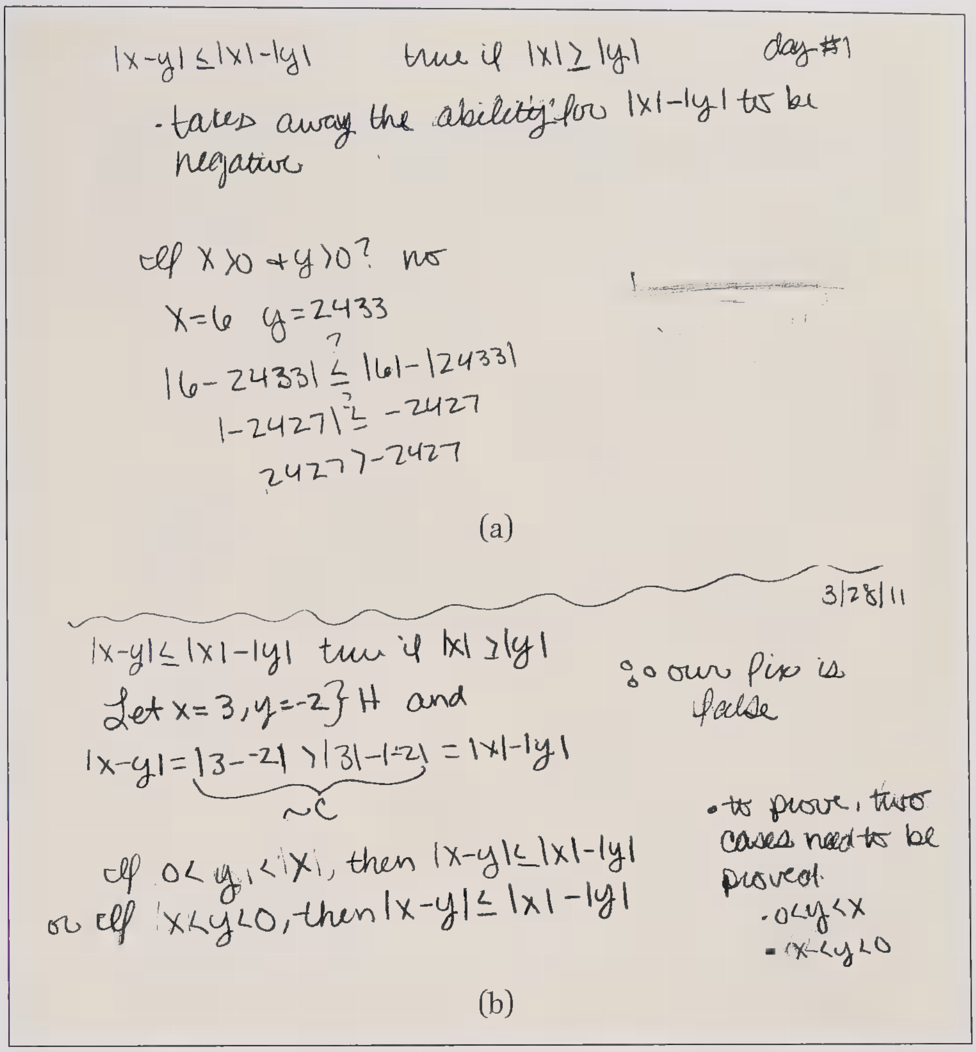
I then asked students to work in pairs or groups of three to form propositions.

SOME PROPOSITIONS

After the discussion, students produced several propositions ("fixes"). In our classroom's vocabulary, these fixes were propositions, not conjectures, because students were asserting that their statements were true. That evening, I sorted the assertions into five categories:

1. If $|x| \geq |y|$, then $|x - y| \leq |x| - |y|$.
2. If $x \geq 0$ and $y \geq 0$, then $|x - y| \leq |x| - |y|$. (One group included the additional hypothesis that $y < x$.)
3. For all real numbers x and y, $|x - y| \geq |x| - |y|$.
4. If $x \geq 0$ and $y \geq 0$, then $|x - y| \geq |x| - |y|$.
5. For all real numbers x and y, $|x - y| \geq ||x| - |y||$.

During the next class period, I wrote all five propositions on the board, and we compared the fixes. We discussed how propositions 1 and 2 alter the hypothesis, how propositions 3 and 5 alter the conclusion, and how proposition 4 alters both. I then asked students to return to their groups (the same in which they had made the propositions) and repeat the original process. They were to either prove their proposition or provide a counterexample. Selected student responses are presented here. The first part of each figure referenced is the students' fix to the original conjecture; the second part is the students' evaluation (proof, counterexample, etc.) of their proposition.


$|x-y| \leq |x|-|y|$ true if $|x| \geq |y|$ day #1

- takes away the ability for $|x|-|y|$ to be negative

if $x > 0$ + $y > 0$? no

$x = 6$ $y = 2433$

$|6-2433| \overset{?}{\leq} |6|-|2433|$

$|-2427| \overset{?}{\leq} -2427$

$2427 > -2427$

(a)

3/28/11

$|x-y| \leq |x|-|y|$ true if $|x| \geq |y|$

Let $x=3, y=-2 \}$ H and

$|x-y| = \underbrace{|3--2| > |3|-|-2|}_{\sim C} = |x|-|y|$

∴ our fix is false

if $0 < |y| < |x|$, then $|x-y| \leq |x|-|y|$

or if $x < y < 0$, then $|x-y| \leq |x|-|y|$

- to prove, two cases need to be proved.
  - $0 < y < x$
  - $x < y < 0$

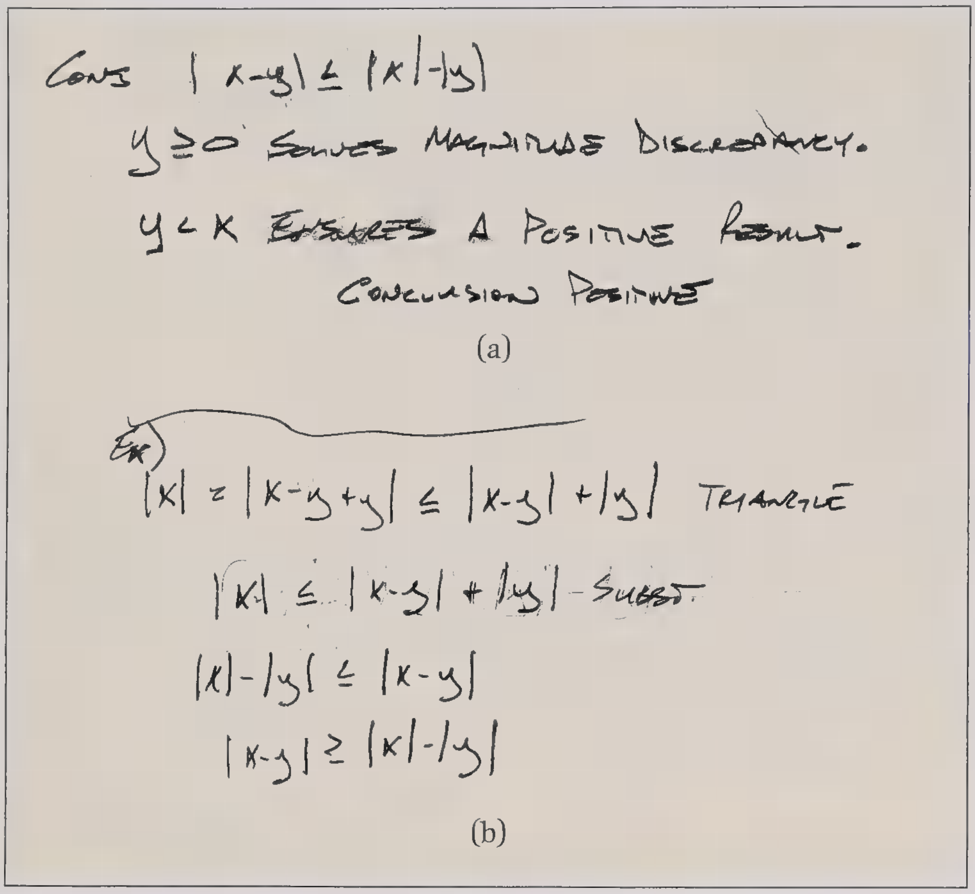
(b)

Fig. 5 Students discover an error (**a**) and try to correct it (**b**).

Ashley and Dakota tell us that their fix attempts to resolve the question of the conjecture's sign (see **fig. 5a**). Then they show us that their fix is false and propose a different fix (see **fig. 5b**). These students are engaging in sense making because they use existing knowledge and prior observations in forming their proposition.

Alex and Casey also engage in sense making by grounding their fix in reasoning about distance (see **fig. 6a**). Then they prove a different result (see **fig. 6b**). We wonder whether they discovered that their fix in **figure 6a**, adding the hypothesis that $0 \leq y < x$, is trivial.

Jamie and Jean make an argument that could qualify as a proof with a few additional words or symbols (e.g., logical connectives and acknowledgment of prior results) (see **fig. 7a**, top left corner). Yet, in the right corner, they attempt another argument that is problematic. In the third line (labeled



Cons $|x-y| \le |x|-|y|$

$y \ge 0$ Solves Magnitude Discrepancy.

$y < x$ ~~Ensures~~ A Positive Result.

Conclusion Positive

(a)

Ex)

$|x| = |x-y+y| \le |x-y| + |y|$ Triangle

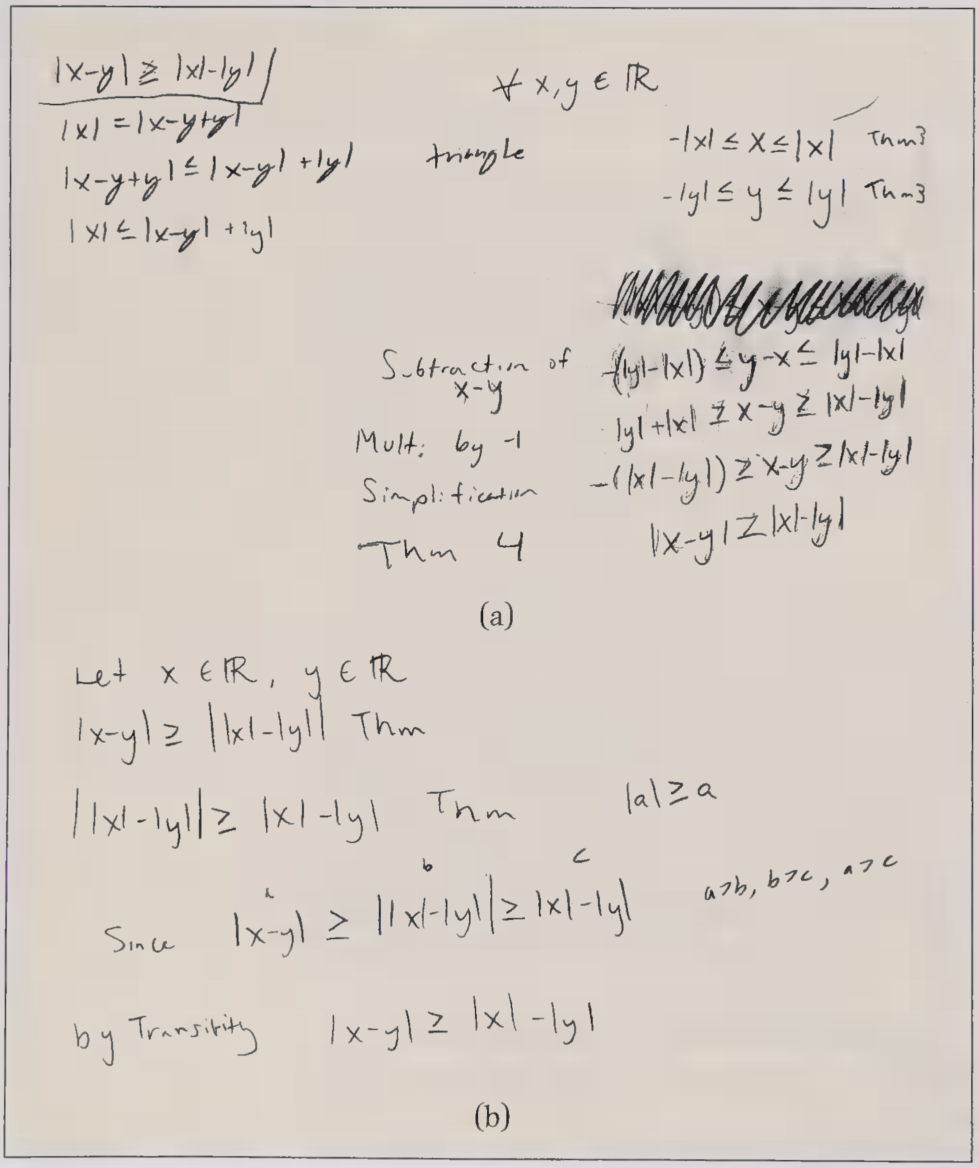
$|x| \le |x-y| + |y|$ - Subst.

$|x|-|y| \le |x-y|$

$|x-y| \ge |x|-|y|$

(b)

Fig. 6 Students use distance (**a**) to produce a different result (**b**).



$|x-y| \ge |x|-|y|$

$|x| = |x-y+y|$

$|x-y+y| \le |x-y| + |y|$ triangle

$|x| \le |x-y| + |y|$

$\forall x,y \in \mathbb{R}$

$-|x| \le x \le |x|$ Thm3

$-|y| \le y \le |y|$ Thm3

[redacted]

Subtraction of $x-y$ $\quad -(|y|-|x|) \le y-x \le |y|-|x|$

Mult. by -1 $\quad |y|+|x| \ge x-y \ge |x|-|y|$

Simplification $\quad -(|x|-|y|) \ge x-y \ge |x|-|y|$

Thm 4 $\quad |x-y| \ge |x|-|y|$

(a)

Let $x \in \mathbb{R}$, $y \in \mathbb{R}$

$|x-y| \ge ||x|-|y||$ Thm

$||x|-|y|| \ge |x|-|y|$ Thm $\quad |a| \ge a$

Since $\overset{a}{|x-y|} \ge \overset{b}{||x|-|y||} \ge \overset{c}{|x|-|y|}$ $\quad a>b, b>c, a>c$

by Transitivity $|x-y| \ge |x|-|y|$

(b)

Fig. 7 Although some of the work in both (**a**) and (**b**) is correct, there is a reasoning flaw in (**a**).

"subtraction of $x - y$"), they misapply or inappropriately generalize the addition property of inequalities to subtraction. Also, the implication from the second-to-last line (labeled "simplification") to the last line is incorrect for two reasons: (1) we cannot assume that $-(|x| - |y|)$ is greater than zero; and (2) the property that the students appear to use, $0 \le |a| \le c \leftrightarrow -c \le a \le c$, is used incorrectly, even if they assume that $|x| - |y|$ is greater than zero.

Jamie and Jean write a third argument (see **fig. 7b**). This argument is correct, and it uses a very strong result in step 2. (This result was proven in class between these two activities.)

Jamie and Jean's arguments generated considerable classroom discussion. Although students valued both arguments, some preferred the last argument because it seems more sophisticated (e.g., it uses a stronger result). However, other students preferred the first argument because, in their words, "it better demonstrates why it is true," and it appeals to basic inequality principles.

The last example of student work presented here (see **fig. 8**) demonstrates how one group that conjectured about the fix in proposition 2 proves a different result (proposition 3) and then reconsiders the fix in an attempt to find a more appropriate hypothesis. Like Ashley and Dakota, Skylar and Jerry also model the intertwining of informal and formal reasoning as they make sense of the situation (NCTM 2009).

The fifth type of proposition listed above was a theorem stated in the text as a conjecture and proven in class while this activity was ongoing. (Jamie and Jean appeal to this result in the third argument for their proposition.) When the students who produced proposition 5 were asked to prove it, they mimicked the proof given in class. I did not criticize this approach; instead, I valued this work because mathematicians often appeal to results that they already know to be true, and they have a large repertoire of proof strategies and known proofs that they use.

REFLECTION

For this final section, I have chosen to write a reflection rather than a conclusion. To call this a conclusion would express intent other than the spirit in which this article is written. My goal as an educator is to develop and implement learning experiences for my students that reflect current understanding of mathematical teaching practice. During this activity, my goal was to facilitate students' learning about the interplay between informal and formal reasoning as well as to help them develop a "mathematical habit of mind" that involves a search for truth when faced with falsehoods. To say that I draw conclusions from this one experience would misrepresent my understanding of how mathematical best practices develop. Instead, I wrote this article to share my practice and to contribute to dialogue about high-quality, reasoning-based learning environments.

I now present some aspects of this activity on which I have reflected. First, I felt that students had an experience that promises to develop their understanding of mathematics as a discipline. We examined a conjecture, proved it false, examined the reasoning behind our counterexample, used this reasoning to develop some propositions, and then examined our propositions for truth values. This process exposed students to the nature of mathematical investigation, which includes examining, investigating, conjecturing, proposing, and reexamining in a continual quest for sense making. Structuring the activity as a sequence of three episodes over a two-week period exposed students to the concept that mathematics is not a lecture-homework-study-exam experience but, instead, a continuous and personalized quest for truth.

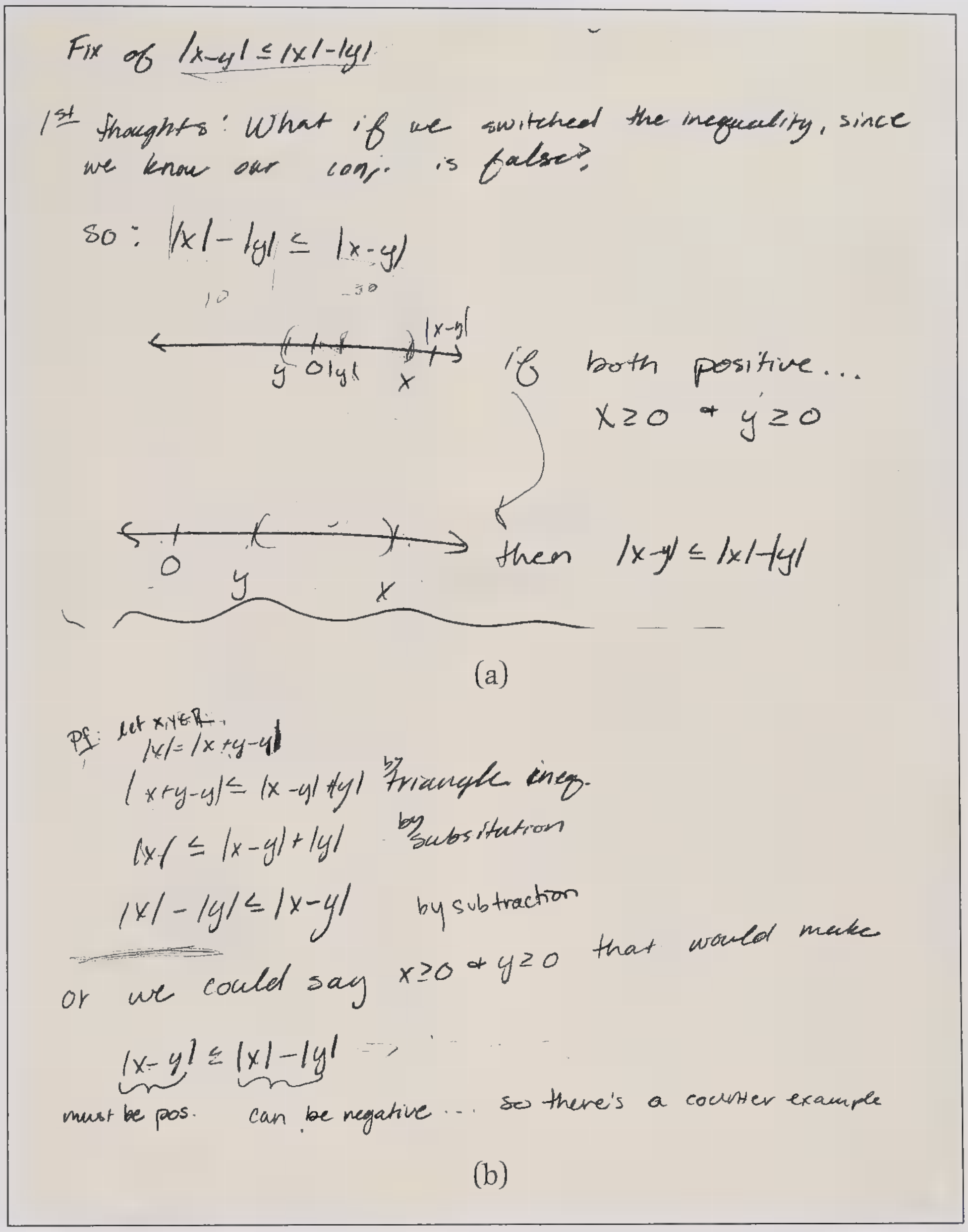
The fact that the activity began with a counterexample is also important. Too often, we demonstrate a counterexample and move on, satisfied that we have evaluated the claim. Logically, this is true; mathematically, however, the work of sense making has just begun. Students should learn that "doing mathematics" often involves learning from false starts.

Students also had opportunities to experience roles of proving and reasoning beyond establishing truth. Counterexamples do not have to explain why. However, students saw that when there is reasoning behind the counterexamples, opportunities for more learning emerge. In this sense, our proving was related to problem solving (e.g., solving the problem of writing a true statement that salvages as much of the initial conjecture as possible).

The reasoning and proving activities also helped students discover new mathematical propositions not in their textbook, and in some cases they even created new mathematics for themselves. They used their counterexamples (proofs of "there exist" statements) to develop and discover a new mathematical statement (e.g., for all x and y, [blank] is true).

Last, through reasoning and proving, the students had opportunities to learn about logic, absolute value properties, and absolute values and real number line distances, and they became more aware and appreciative of the work of mathematicians.

Students also observed the value of multiple solutions in light of their individual contribution to our mathematical understanding at every stage of the endeavor. Different counterexamples arose from different types of reasoning, which led to very different fixes, and fixing false conjectures involved altering hypotheses, conclusions, or both.



Fix of $|x-y| \le |x|-|y|$

1st thoughts: What if we switched the inequality, since we know our conj. is false?

so: $|x|-|y| \le |x-y|$

if both positive...
$x \ge 0$ + $y \ge 0$

then $|x-y| \le |x|-|y|$

10
30
|x-y|
y
0
|y|
x
0
y
x

(a)

Pf: let $x, y \in \mathbb{R}$.
$|x| = |x+y-y|$
$|x+y-y| \le |x-y| + |y|$ by triangle ineq.
$|x| \le |x-y| + |y|$ by substitution
$|x| - |y| \le |x-y|$ by subtraction

or we could say $x \ge 0$ + $y \ge 0$ that would make

$\underbrace{|x-y|}_{\text{must be pos.}} \le \underbrace{|x|-|y|}_{\text{can be negative}}$ ... so there's a counter example

(b)

Fig. 8 Students whose work is shown in (**a**) and (**b**) use both formal and informal reasoning to prove proposition 3.

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JUNE

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

For how many three-digit numbers between 100 and 999 do the first and last digits differ by 3?

1

Exactly three of the numbers in the set

$$\{12, 14, 15, 20\}$$

divide evenly into some number n. Which number will not divide evenly into n?

2

Calculate the average of the four prime numbers closest to 58.

3

What is the largest number that will divide every number greater than 1000 that ends in 768?

4

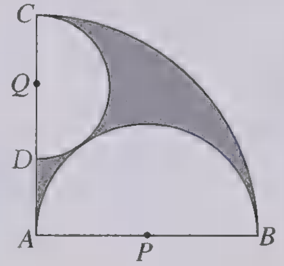
If

$$2a + 5a + 8a + 11a = \frac{b}{2} + \frac{b}{5} + \frac{b}{8} + \frac{b}{11},$$

find the ratio of a to b.

5

Arc BC is a quarter circle with radius 2 and center A. P and Q are the centers of mutually tangent semicircles with diameters AB and CD, respectively. Compute the shaded area.



6

Which of the following is larger?

$$\sqrt[99]{99!}$$

or

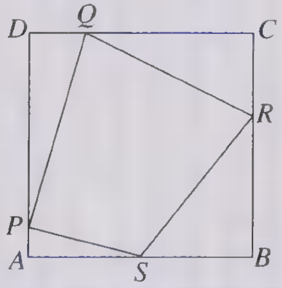
$$\sqrt[100]{100!}$$

7

For how many four-digit numbers does the sum of the digits equal 9?

8

Given square $ABCD$ with side length 8, find the area of $PQRS$ if $AP = 1$, $DQ = 2$, $CR = 3$, and $BS = 4$.



9

Solve the following:

$$3^{2x+1} - 3^{x+3} - 3^{x} + 3^{2} = 0$$

10

The numeral 65 in base x represents the same number as 56 in base y. Find the smallest value of $x + y$.

11

Eleven toothpicks can be arranged to form the following incorrect statement involving roman numerals. Move just one toothpick to make the equation correct. (There are several solutions.)

VI + II = V

In an orchestra, 23 people can play stringed instruments, 18 can play brass, and 12 can play percussion. Further, 10 of the performers can play both strings and brass, whereas 6 can play both strings and percussion. If no one can play all three types of instruments, what are the maximum and minimum numbers of people in

A salesman took a $20 taxi ride to get to his first client, but there he tripled the amount of money he had left. His ride to the second client cost $12, and there he only doubled what he had left. The cost of the ride to his third client was $17, and again he doubled the amount of money he had. If the salesman ended with $626, how

A square and an equilateral triangle have the same perimeter. Compute the ratio of the area of the triangle to the area of the square.

A square and an equilateral triangle have equal areas. Compute the ratio of the perimeter of the triangle to the perimeter of the square.

16

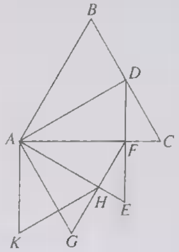
Square *ABCD* has two quarter circles drawn in its interior. The centers of the circles are two consecutive vertices of the square, and the radius of each circle is the length of a side of the square. If the side of the square is 8, what is the distance from the intersection of the quarter circles to the nearest side of the square?

17

Find the sum of all three-digit integers whose units digit is 7.

18

$\triangle ABC$ is equilateral with altitude AD. Use the altitude to form a second equilateral triangle, $\triangle ADE$. Use the altitude of $\triangle ADE$ to form a third equilateral triangle, $\triangle AFG$, and then use the altitude of $\triangle AFG$ to form a fourth equilateral triangle, $\triangle AHK$. Find the ratio of the area of $\triangle ABC$ to the area of $\triangle AHK$.



19

How many different types of cross sections can be made by cutting a cube with a plane? Two examples are a square and an equilateral triangle.

20

Find the minimum value of the following expression:

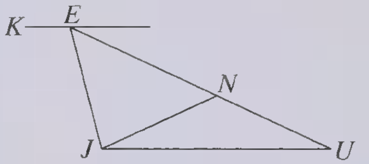
$$|x-3|+|x-1|+|x|+|x+2|+|x+4|$$

21

Three circles—*P*, *Q*, and *R*—are externally tangent to one another. The radii of the circles are 5, 12, and 13, respectively. Name the smallest angle in $\triangle PQR$.

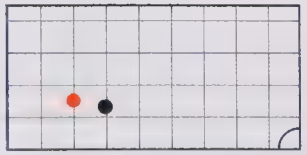
22

In $\triangle EJU$, $m\angle NJU = 25°$, and $m\angle JNU = 130°$. If $\overline{KE} \parallel \overline{JU}$ and $NU = EJ$, find $m\angle KEJ$.



23

A coordinate grid is placed on a 4.5 × 9-ft. pool table. The object is to sink the ball located at (2, 1.5) in the hole located at (9, 0). Since a direct shot is blocked, the shooter must bank her shot off an edge opposite the hole. Find the coordinates of the point she must hit to make the shot.



24

Find two numbers *a* and *b* such that

$$(a-b):(a+b):ab = 1:5:24.$$

25

The line $y = mx + 3$ is tangent to the ellipse whose equation is

$$4x^2 + y^2 = 4.$$

Find the value of m^2.

26

Find the ordered pair of integers (x, y) that satisfy the equation

$$3^{2x} - 2^{2y} = 17.$$

27

The coordinates of three points are $A(-3, 2)$, $B(25, 10)$, and $C(k, k)$. Find the value of *k* such that $AC + BC$ is a minimum.

28

The coordinates of three points are $A(-3, 2)$, $B(5, 15)$, and $C(k, k)$. Find the value of *k* such that $AC + BC$ is a minimum.

29

Solve the following equation for *x*:

$$\sqrt{x+\sqrt{x+271}}+\sqrt{x-\sqrt{x+271}}=6$$

30

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1. 130. Consider a pair of digits whose difference is 3, such as 4 and 1. For this pair, there are 10 numbers with 1 in the hundreds place and 10 numbers with 4 in the hundreds place. Six such pairs exist: (1, 4), (2, 5), (3, 6), (4, 7), (5, 8), and (6, 9). The pair (3, 0) can be included but not (0, 3). Therefore, there are $20 \cdot 6 + 10 = 130$ numbers.

2. 14. Consider the factors of each number in the set: $12 = 2^2 \cdot 3$, $14 = 2 \cdot 7$, $15 = 3 \cdot 5$, and $20 = 2^2 \cdot 5$. Therefore, 12, 15, and 20 will divide $3 \cdot 4 \cdot 5$, and 14 will not divide 60.

3. 60. The four prime numbers closest to 58 are 53, 59, 61, and 67. The sum of these four numbers is 240, so their average is 60.

4. 8. We check for divisibility by 2 by seeing if the units digit in the number is divisible by 2, because no matter how many tens we have, they will always be divisible by 2. Similarly, since 100 is divisible by 4, if a number ends in two digits that are divisible by 4, then the entire number must be divisible by 4. Since 768 is divisible by 8 ($768/8 = 96$), and 1000 is divisible by 8, any number greater than 1000 and ending in 768 is divisible by 8.

5. 31/880. Sum the numbers on each side of the equation to get $26a = 403b/440$. Then divide both sides of the equation by $26b$.Thus, $a/b = 403/(440 \cdot 26) = 31/880$.

6. $5\pi/18$. The area of the quarter circle is $4\pi/4 = \pi$. The area of the larger semicircle is $\pi/2$. Let r be the radius of the smaller semicircle. Then $AQ = 2 - r$, and $PQ = 1 + r$. Apply the Pythagorean theorem to $\triangle AQP$ and solve for r: $(2 - r)^2 + 1^2 = (1 + r)^2 \rightarrow 4 - 4r + r^2 + 1 = r^2 + 2r + 1 \rightarrow r = 2/3$. Thus, the area of the small semicircle is $(1/2)(4\pi/9) = 2\pi/9$. Finally, the area of the shaded portion is $\pi - (\pi/2 + 2\pi/9) = \pi - 13\pi/18 = 5\pi/18$.

7. $\sqrt[100]{100!}$. Most calculators would have trouble computing these values, so we need to approach this problem in a different way. Consider $100! = 100 \cdot 99!$ and then eliminate the roots by raising both expressions to the 9900th power. These operations result in $(99!)^{100} = (99!)(99!)^{99}$ and $(100 \cdot 99!)^{99} = 100^{99} \cdot (99!)^{99}$. Now it should be clear that $100^{99} > 99!$. So $\sqrt[100]{100!} > \sqrt[99]{99!}$. Surprisingly, the two given values differ by only about 0.36958887, even though $100^{99} - 99! \approx 10^{198}$.

8. 165. Making a list produces some interesting patterns. One pattern results from fixing the first digit and counting the remaining possibilities. For example, when the first digit is 9, only one number can be formed: {9000}. When the first digit is 8, three numbers are possible: {8100, 8010, and 8001}. When the first digit is 7, six numbers can be formed: {7110, 7101, 7001, 7200, 7020, and 7002}. When the first digit is 6, the set of numbers contains 10 elements {6300 can be rewritten in three ways, 6210 in six ways, and 6111 in only one way}. The numbers 1, 3, 6, and 10 are the first four triangular numbers—that is, numbers formed by adding consecutive integers. Therefore, the next few results will be $10 + 5 = 15$, $15 + 6 = 21$, $21 + 7 = 28$, then 36, and finally 45. The sum of all of these numbers is 165.

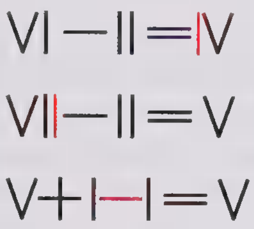
Alternate solution: We can apply the following reasoning based on the multinomial theorem. Consider 9 dots in a row. Three dashes are required to separate them into 4 groups. Since there are 12 objects (dots and dashes) and we want to arrange them every way possible, there are ${}_{12}C_3$ ways to do that. Since we cannot begin a four-digit number with a zero, we must subtract ${}_{11}C_2$ from ${}_{12}C_3$ to end with ${}_{12}C_3 - {}_{11}C_2 = 220 - 55 = 165$.

9. 36. Subtract the areas of the four right triangles from the area of the square: $8 \cdot 8 - (1/2)(7 \cdot 2 + 6 \cdot 3 + 5 \cdot 4 + 4 \cdot 1) = 64 - (1/2)(14 + 18 + 20 + 4) = 64 - 28 = 36$.

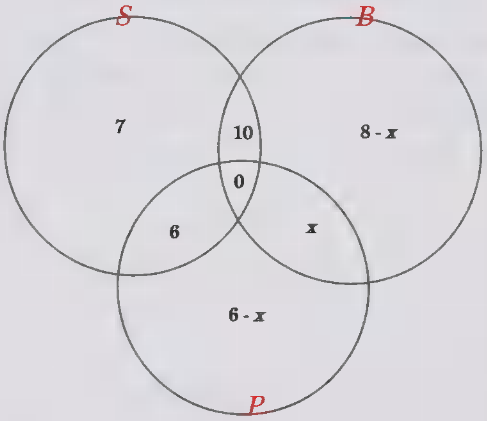
10. −1 or 2. Let 3^x be represented by y. Then the equation becomes $3y^2 - 28y - y + 9 = 0$. This is a quadratic equation in y. Factor and solve: $(3y - 1)(y - 9) = 0 \rightarrow y = 1/3$ or $y = 9 \rightarrow x = -1$ or 2.

11. 24. Since $6x + 5 = 5y + 6$ and x and y are both greater than 6, we need to find integer solutions to the equation $y = (6/5)x - 1/5$. One ordered pair that satisfies the equation is (1, 1). This pair does not satisfy the minimum condition, so we can apply the notion of slope to find other pairs of numbers. The change in y is 6 for every change of 5 in x, so the next pairs are (6, 7) and (11, 13). The first pair to satisfy the minimum restriction is (11, 13) and $11 + 13 = 24$.

12. This problem has at least three possible solutions:



13. The maximum is 37, and the minimum is 31. Use a Venn diagram to examine the data. Since there are 10 and 6 performers, respectively, in the regions common to string and brass and to string and percussion, there must be 7 in string alone. Let x represent the number common to brass and percussion. Therefore, there are $8 - x$ in brass alone and $6 - x$ in percussion alone. The sum of all the nonoverlapping regions is $37 - x$ where $0 \le x \le 6$. Thus, the maximum occurs when $x = 0$, and the minimum occurs when $x = 6$.



14. $79. This problem can be solved by working backward. The salesman ended with $626, meaning that he had half that amount plus $17, or $330, when he left the second client. When he left the first client, he had half of $330 plus $12: $177. When he left for the first client, he had one-third of $177 plus $20: $79.

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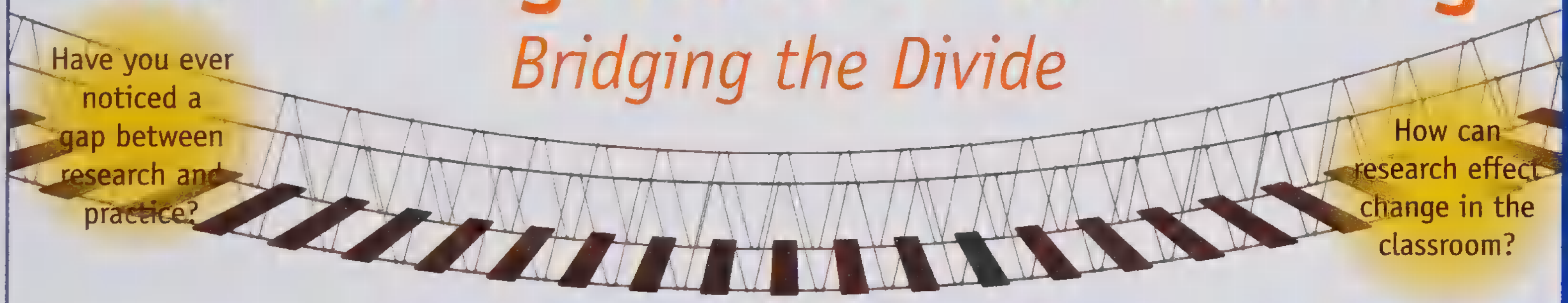

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MATHEMATICS teacher

Alternate solution: Let x = the starting amount. Then solve the equation $2(2(3(x - 20) - 12) - 17) = 626$.

15. $\frac{4\sqrt{3}}{9}$. Let s be a side of the square and let t be a side of the triangle. Then $4s = 3t$, and thus $s = 3t/4$. The area of the triangle is $t^2\sqrt{3}/4$, and the area of the square is $s^2 = 9t^2/16$. Therefore, the ratio of the areas is

$$\frac{\frac{t^2\sqrt{3}}{4}}{\frac{9t^2}{16}} = \frac{t^2\sqrt{3}}{4} \bullet \frac{16}{9t^2} = \frac{4\sqrt{3}}{9}.$$

16. $\frac{\sqrt[4]{27}}{2}$. Let s be the side of the square and let t be the side of the triangle. Then

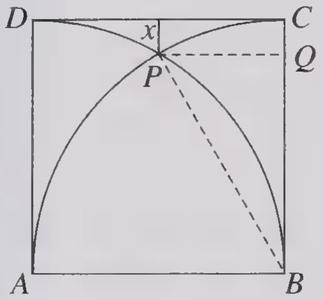
$$s^2 = \frac{t^2\sqrt{3}}{4} \rightarrow$$

$$4s^2 = t^2\sqrt{3} \rightarrow t = \frac{2s}{\sqrt[4]{3}}.$$

To find the ratio, we compute as follows:

$$\frac{3t}{4s} = \frac{6s}{\sqrt[4]{3}} \bullet \frac{1}{4s} = \frac{3}{2 \bullet 3^{\frac{1}{4}}} = \frac{3^{\frac{3}{4}}}{2} = \frac{\sqrt[4]{27}}{2}$$

17. $8 - 4\sqrt{3}$. Let x represent the distance we wish to find. Construct a perpendicular segment from the intersection P to $\overline{BC}$ and call the new point Q. Draw a segment from B to P. Then $\triangle BPQ$ is a right triangle with $BP = 8$ and $PQ = 4$. Either recognize that the triangle is a 30°-60°-90° triangle or apply the Pythagorean theorem to find that $BQ = 4\sqrt{3}$. Finally, $x = CQ = BC - CQ = 8 - 4\sqrt{3}$.



Alternate solution: Create the diagram on a grid with the lower-left corner of the square at the origin. The equations of the circles would be $x^2 + y^2 = 64$ and $(x - 8)^2 + y^2 = 64$. Solve the system of equations by subtracting the first equation from the second to get $-16x + 64 = 0$. Then $x = 4$, and $y = \sqrt{48} = 4\sqrt{3}$. The solution is therefore $8 - 4\sqrt{3}$.

18. 49,680. The numbers 107 + 117 + 127 + ⋯ + 997 form an arithmetic series. Because every tenth number ends in a 7 and there are 900 numbers from 100 to 1000, there are 90 terms in the series. The formula for the sum of an arithmetic series is $S = n(a_1 + a_n)/2 = 90(107 + 997)/2 = 49{,}680$.

19. 64/27. The length of the altitude of an equilateral triangle with side s is $s\sqrt{3}/2$. Repeating this process, we find that the lengths of the sides of the equilateral triangles form the sequence s, $s\sqrt{3}/2$, $3s/4$, and $3s\sqrt{3}/8$. The ratio of the areas of similar figures is the ratio of the squares of a pair of corresponding sides—in this case,

$$\left(\frac{1}{\frac{3\sqrt{3}}{8}}\right)^2 = \left(\frac{8}{3\sqrt{3}}\right)^2 = \frac{64}{27}.$$

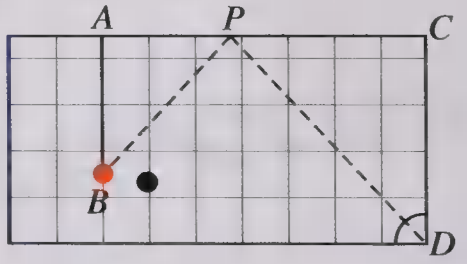
20. 7. The different cross sections that can be formed are the square, equilateral triangle, rectangle (other than the square), nonequilateral triangle, pentagon, hexagon, and parallelogram (other than a rectangle). (See http://www.learner.org/courses/learningmath/geometry/session9/part_c/index.html.)

21. 10. The graph of $y = |x - 3| + |x - 1| + |x| + |x + 2| + |x + 4|$ is a set of line segments that approximates a parabola. When $x < -4$ or $x > 3$, each addend grows greater in absolute value, so the minimum must be between those two numbers. Consider creating a table of values for $x = -4, -2, 0, 1$, and 3. Each makes one of the terms in the expression zero. The corresponding results are 18, 12, 10, 11, and 17. Now look at the expression more carefully for values of x between -2 and 1 where there appears to be a minimum. The line segment from -2 to 0 has a slope of -1, and the line segment from 0 to 1 has a slope of 1. Therefore, the minimum will be where the two segments intersect—that is, when $x = 0$ and $y = 10$. (Note: Standard deviation deals with the quadratic expression used to estimate the mean absolute deviation. This quadratic is based on the sum of the absolute values of the differences between the mean and each individual value.)

22. Angle R. Adding pairs of radii determines the lengths of the sides of the triangle. Therefore, $PQ = 5 + 12 = 17$; $QR = 12 + 13 = 25$; and $RP = 13 + 5 = 18$. The smallest angle in a triangle is opposite the shortest side, so $\angle R$ is the smallest angle.

23. 105°. The sum of the angles of a triangle is 180°, so $m\angle NUJ = 25°$. Since two angles of $\triangle JUN$ are congruent, $NU = NJ$. Then $m\angle JNE = 50°$; and since $EJ = NJ$, $m\angle NEJ$ is also 50°, resulting in $m\angle EJN = 80°$. Angles KEJ and EJU are congruent alternate interior angles for the parallel lines, so $m\angle KEJ = 80° + 25° = 105°$.

24. (4.8, 4.5). Point P is the point to aim for. This point will make the angle of incidence equal the angle of reflection—that is, $\angle BPA = \angle DPC$. To find the coordinates of P, recognize that $\triangle BPA \sim \triangle DPC$, so $AB/AP = CD/CP \rightarrow 3/x = 4.5/(7 - x)$. Solve by multiplying by $x(7 - x)$ to get $21 - 3x = 4.5x \rightarrow x = 2.8$. The coordinates of P are (4.8, 4.5).



25. $a = 12$ and $b = 8$. Work with the two equations $(a - b)/(a + b) = 1/5$ and $(a + b)/(ab) = 5/24$. Solve the first equation for a to get $a = 3b/2$ and substitute this into the second equation. Therefore, $(3b/2 + b)/(3b^2/2) = 5/24 \rightarrow (5b/2)/(3b^2/2) = 5/24 \rightarrow b = 8$. Substitution yields $a = 3(8)/2 = 12$.

26. 5. Since the line intersects the ellipse, we can substitute the expression for y into the equation of the ellipse. The result is $4x^2 + (mx + 3)^2 = 4 \rightarrow 4x^2 + m^2x^2 + 6mx + 9 = 4 \rightarrow (4 + m^2)x^2 + 6mx + 5 = 0$. Since the line can intersect the ellipse at only one point, this quadratic equation must have only one solution, implying that its discriminant must be zero. Thus, we have $b^2 - 4ac = 36m^2 - 4(5) \cdot (4 + m^2) = 0 \rightarrow 16m^2 = 80 \rightarrow m^2 = 5$.

27. (2, 3). Rewrite the equation as $(3^x)^2 - (2^y)^2 = 17$. Factor the left side as the difference of two perfect squares: $(3^x - 2^y)(3^x + 2^y) = 17$. Since $(3^x + 2^y) > (3^x - 2^y)$ and the only integer factors of 17 are 1 and 17 or −1 and −17, we have $3^x + 2^y = 17$ or −1. We reject −1 because exponential expressions are always positive. There are two solutions to the equation $3^x + 2^y = 17$: (0, 4) and (2, 3). Substituting these into the equation $3^x - 2^y = 1$ yields the only solution of (2, 3).

Alternate solution: Trial and error works nicely here as well because we are working with integers. Look for even powers of 3 that are 17 more than an even power of 2.

28. 4. Notice that the line $y = x$ intersects $\overleftrightarrow{AB}$ between A and B. Therefore, the minimum of $AC + BC$ will occur when C is that point of intersection and the three points are collinear. Set the slopes of the segments determined by two pairs of points equal and solve for k: $(k - 2)/(k + 3) = (10 - 2)/(25 + 3) = 2/7 \rightarrow 7k - 14 = 2k + 6 \rightarrow 5k = 20 \rightarrow k = 4$.

29. 3. Although it appears similar, this problem is more difficult than the previous one. In this case, the line $y = x$ does not intersect $\overleftrightarrow{AB}$ between A and B. The solution lies on $y = x$, but where? To find the coordinates of this point, let y = the sum of the distances using the distance formula. Therefore,

$$y = \sqrt{(k+3)^2 + (k-2)^2} + \sqrt{(k-5)^2 + (k-15)^2}.$$

To find the minimum, replace k with x and graph y on a calculator. Use the minimum function to determine that $x = 3$. The sum of the distances is $3\sqrt{37}$.

Alternate solution: Reflect B over the line $y = x$ to find B' (15, 5). The line AB' will intersect $y = x$ at a point P, thus minimizing the sum of the distances since $PB' = PB$. To find the coordinates of P, find the equation of AB': slope of $AB' = (5 - 2)/(15 + 3) = 1/6$, so the equation is $y - 2 = (x + 3)/6$. When $y = x$, $x - 2 = (x + 3)/6 \rightarrow 6x - 12 = x + 3 \rightarrow x = 3$ and $y = 3$. Apply the distance formula to obtain $AP + PB = 3\sqrt{37}$.

30. 17. Square both sides of the equation to obtain the following:

$$\begin{aligned} x + \sqrt{x+271} + 2\sqrt{x^2 - x - 271} \\ + x - \sqrt{x+271} = 36 \rightarrow \\ 2x + 2\sqrt{x^2 - x - 271} = 36 \rightarrow \\ \sqrt{x^2 - x - 271} = 18 - x \end{aligned}$$

Squaring both sides produces $x^2 - x - 271 = 324 - 36x + x^2 \rightarrow 35x = 595 \rightarrow x = 17$.

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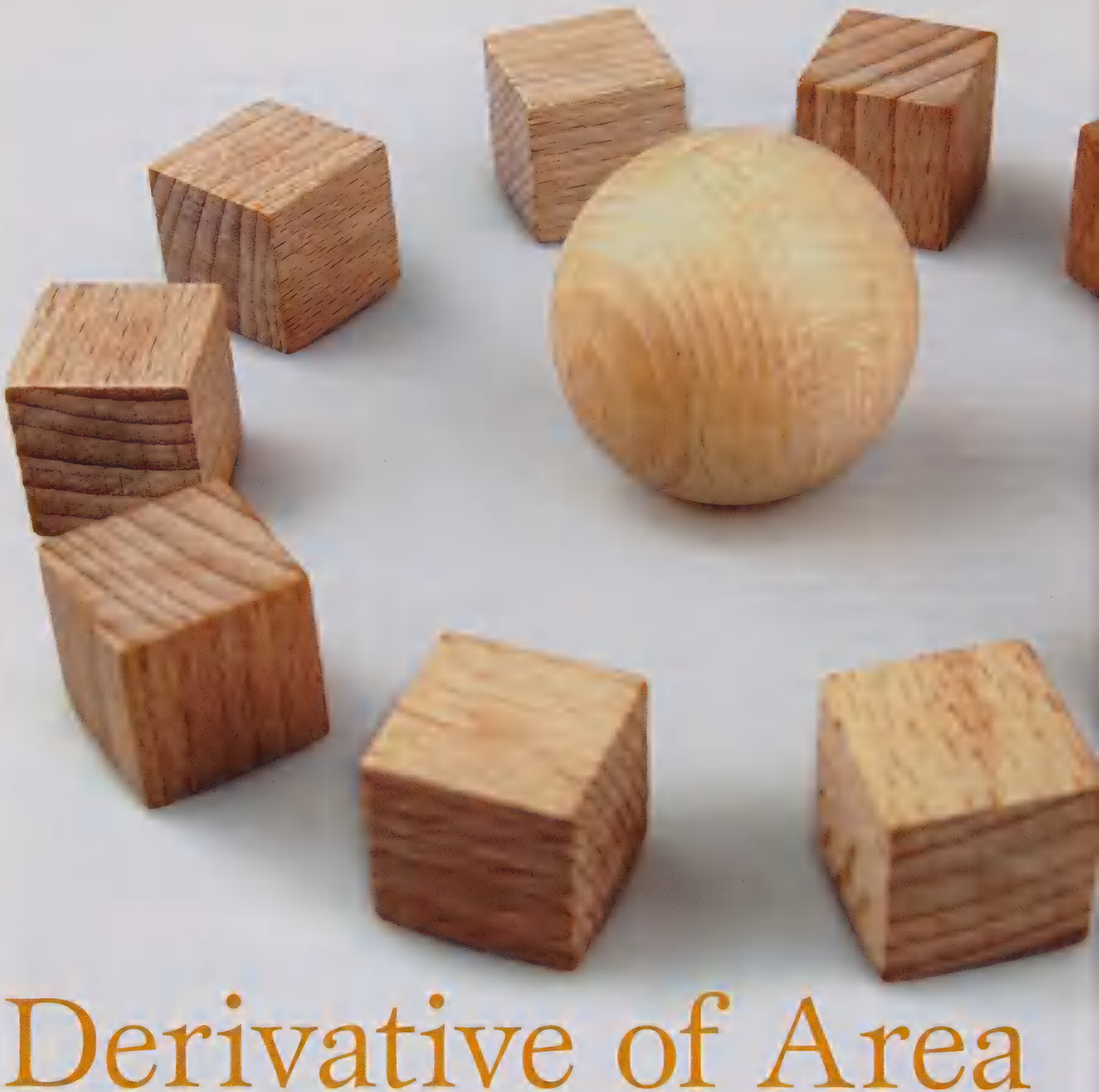


Derivative of Area Equals Perimeter—Coincidence

or Rule ?

A familiar relationship—the derivative of the area of a circle equals its circumference—is extended to other shapes and solids.

Rina Zazkis, Ilya Sinitsky, and Roza Leikin

Why is the derivative of the area of a circle equal to its circumference? Why is the derivative of the volume of a sphere equal to its surface area? And why does a similar relationship not hold for a square or a cube? Or does it?

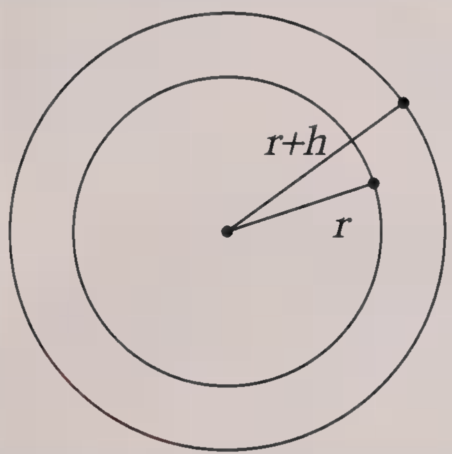
In our work in teacher education, we have heard at times undesirable responses to these questions: "That's the way it is. Circles and spheres are very special. Squares and cubes have corners." Or, "It is a simple coincidence with circles. This relationship does not hold for any other shapes."

In this article, we explore and explain the familiar relationship of the area of a circle and its circumference and of the volume of a sphere and its surface area. We then extend this relationship to other two- and three-dimensional figures—squares and regular polygons, cubes and regular polyhedra.

CIRCLES

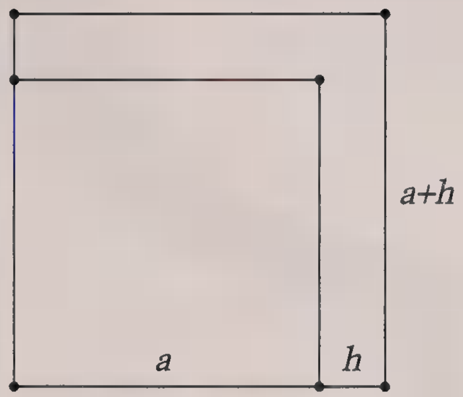
The relationship between the area A of a circle with radius r and its circumference P can be expressed by $A'(r) = (\pi r^2)' = 2\pi r = P(r)$. We call a relationship of this kind a *derivative relationship* between the area and the circumference or perimeter of a figure.

In the case of a circle, the derivative relationship is best seen geometrically (see **fig. 1**). When the radius of the circle is extended from r to $r + h$, the area of the circle changes by $\Delta A = 2\pi rh + \pi h^2$, which is the area of the ring of width h around the circle of radius r. When h approaches zero, the ring approaches the inner circle.



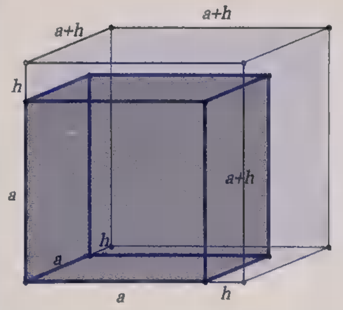
$$A'(r) = \lim_{h\to 0}\frac{\pi(r+h)^2 - \pi r^2}{h} = \lim_{h\to 0}(2\pi r + \pi h) = 2\pi r$$

Fig. 1 The definition of the derivative shows the radius of a circle r extended by h.



$$A'(a) = \lim_{h\to 0}\frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0}(2a + h) = 2a = \frac{P(a)}{2}$$

Fig. 2 Two adjacent sides of the square are each extended by h.



$$V'(a) = \lim_{h\to 0}\frac{(a+h)^3 - a^3}{h} = \lim_{h\to 0}(3a^2 + 3ah + h^2) = 3a^2 = \frac{A(a)}{2}$$

Fig. 3 Three adjacent edges of the cube are each extended by h.

SPHERES

The derivative relationship between the volume of a sphere V and its surface area A is expressed by

$$V'(r) = \left(\frac{4}{3}\pi r^3\right)' = 4\pi r^2 = A(r)$$

because

$$V'(r) = \lim_{h\to 0}\frac{\frac{4}{3}\pi(r+h)^3 - \frac{4}{3}\pi r^3}{h} = \lim_{h\to 0}\left(4\pi r^2 + 4\pi rh + \frac{4}{3}\pi h^2\right) = 4\pi r^2 = A(r).$$

An explanation similar to the explanation given for a circle can be provided: The layer between the outer and inner spheres approaches the inner sphere as h approaches zero. Similar geometric reasoning is found in Usiskin et al. (2003), explaining that this derivative relationship is not coincidental.

We now turn to other shapes.

SQUARES

A square with side a does not fulfill the derivative relationship between the area A and the perimeter P: $A'(a) = (a^2)' = 2a \neq P(a) = 4a$. However, $A'(a) = P(a)/2$.

This statement may initially be considered inconsistent with the derivative relationship described for the circle. The geometrical explanations used above for increasing the side of a square by h demonstrate the geometrical sense of the derivative. As shown in **figure 2**, two adjacent sides of the square are each extended by h. As h approaches zero, the difference of the areas of the two squares tends to half the square's perimeter.

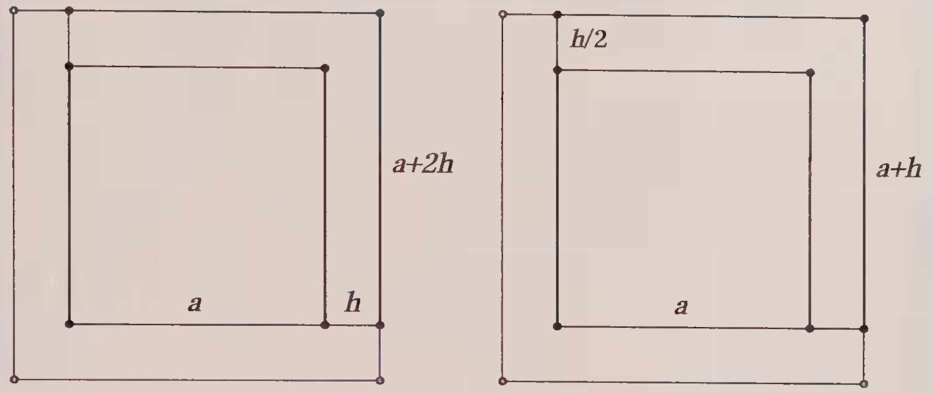
CUBES

A similar "problem" can be seen in the derivative relationship between the volume of a cube with edge a and its surface area. A cube with edge a does not fulfill the derivative relationship between the volume V and the surface area A. This is illustrated in **figure 3**.

An explanation, similar to the one provided for the lack of derivative relationship in a square, can be applied to a cube: Consider a larger cube that results from extending three adjacent edges of the given cube by h (see **fig. 3**). The difference in the volumes of the two cubes, as h approaches zero, tends to half the cube's surface area.

ANALOGY WITH A CIRCLE

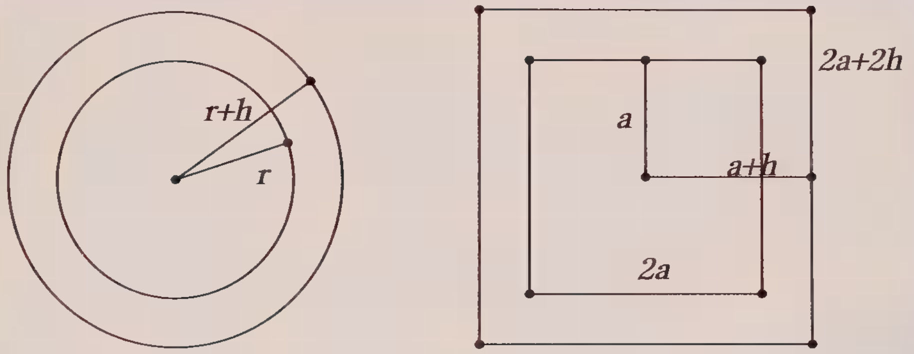
As we seek an analogy between the cases of squares and circles, the obvious attempt is to increase symmetrically all four sides of the square, extending the square in all four directions. That is, to increase the length of each side by h is to add $h/2$ in two directions. Alternatively, if we extend each side of the square by h in two directions, then the length of the side is actually extended by $2h$. However, in these two cases we again find that $A'(a) = P(a)/2$ (see **fig. 4**). So the derivative relationship "fails" again for the square.



$$A'(a) = \lim_{2h \to 0} \frac{(a+2h)^2 - a^2}{2h} = \lim_{2h \to 0}(2a+2h) = 2a = \frac{P(a)}{2}$$

$$A'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0}(2a+h) = 2a = \frac{P(a)}{2}$$

Fig. 4 Each side of a square is extended in two directions (by h or by $h/2$).



For the extended square, $A(a + h) = (2a + 2h)^2 = 4(a + h)^2$ and $P(a) = 4 \cdot 2a = 8a$. Then

$$A'(a) = \lim_{h \to 0} \frac{4(a+h)^2 - 4a^2}{h} = \lim_{h \to 0}(8a+4h) = 8a = P(a).$$

Fig. 5 The analogy between a circle and a square is attained by using side dimensions of $2a$.

Is it possible to achieve consistency?

Let's consider a square with side length $2a$ (see **fig. 5**). An argument analogous to the case of the circle is this: Measuring a as the distance from the center of the square to its side is analogous to measuring the radius of a circle. This makes the length of each side $2a$. This analogy is explored in Movshovitz-Hadar and Webb (1998) and is presented as an important component of teachers' knowledge in Zazkis and Mamolo (2011).

A square with side $2a$ fulfills the derivative relationship between the area A and the perimeter P (see **fig. 5**). For the initial square, $A(a) = (2a)^2 = 4a^2$ and $P(a) = 4 \cdot 2a = 8a$.

Similarly, if the edge of a cube is $2a$, we get the desired derivative relationship, which is analogous to the sphere relationship: The volume of the cube is then $V(a) = (2a)^3 = 8a^3$.

The derivative of the volume of the cube can be expressed as

$$V'(a) = \lim_{h \to 0} \frac{8(a+h)^3 - 8a^3}{h} = \lim_{h \to 0}(24a^2 + 24ah + 8h^2) = 24a^2 = A(a),$$

where $A(a) = 6(2a)^2$ is the surface area of a cube with edge length $2a$.

A cube with edge $2a$ thus fulfills the derivative relationship between the volume V and the surface area A. Thus, the consistency is also obtained for a cube.

HEXAGONS AND OTHER REGULAR POLYGONS

Let's consider other regular polygons. First, let's consider a hexagon with side a (see **fig. 6**). Here, obviously, the derivative relationship between the area and the perimeter does not hold.

As such, our natural question is similar to our initial attempts in considering a square: Is it possible to achieve the derivative relationship for a regular hexagon?

To achieve the derivative relationship for a square, we expressed its area and perimeter through the variable a, where a was half the length of the square's side, which also equals the height of each of the four triangles composing the square (see **fig. 7**). By analogy, we may express the area and perimeter of the hexagon in terms of the height k of an equilateral triangle, six of which compose the hexagon (see **fig. 7**).

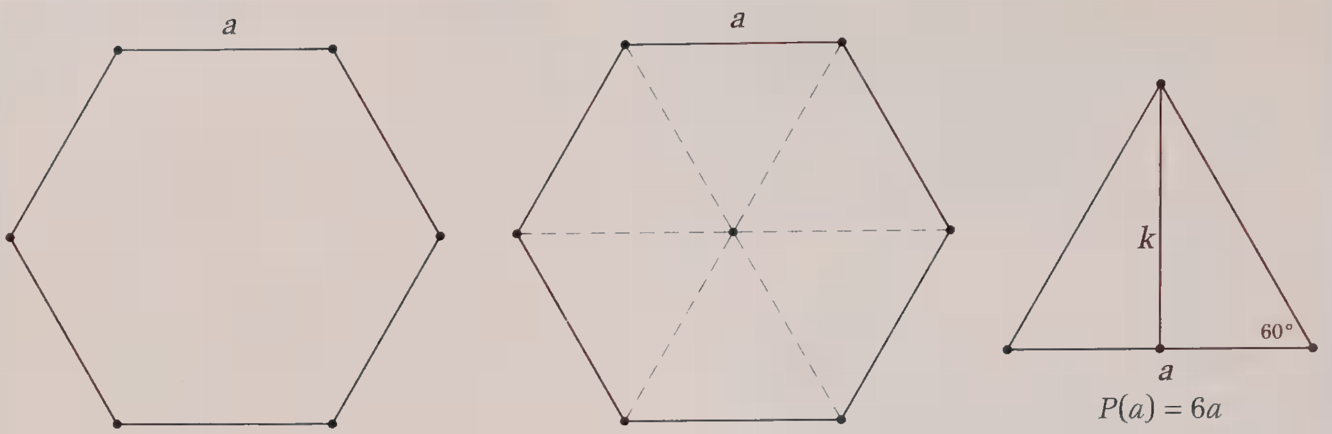
In the initial hexagon, $P(k) = 6a = 4k\sqrt{3}$, so

$$A(k) = \frac{3a^2\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}\left(\frac{2k}{\sqrt{3}}\right)^2 = 2\sqrt{3}k^2 \rightarrow$$

$$A'(k) = \lim_{h \to 0} \frac{2\sqrt{3}(k+h)^2 - 2\sqrt{3}k^2}{h} = 4k\sqrt{3} = P(k).$$

A regular hexagon fulfills the derivative relationship between the area A and the perimeter P: $A'(k) = P(k)$ where k is the radius of the inscribed circle.

In the case of a hexagon as well as in case of a square, the measure (the height of a triangle) according to which the derivative relationship



For each of the six equilateral triangles with side a and height k that compose the hexagon,

$$A_\Delta(a) = \frac{ak}{2} = \frac{a \bullet a\frac{\sqrt{3}}{2}}{2} = \frac{a^2\sqrt{3}}{4};$$

the area of a hexagon is thus

$$A(a) = 6\frac{a^2\sqrt{3}}{4} = \frac{3\sqrt{3}a^2}{2}$$
$$A'(a) = 3\sqrt{3}a \neq P(a).$$

Fig. 6 Testing derivative relationship for a regular hexagon with side *a* results in disappointment.

between area and perimeter exists is the radius of the inscribed circle. This yet again reinforces the analogies with the derivative relationship of a circle (see **fig. 8**).

GENERALIZING THE RESULTS

Now an obvious generalization follows. For any regular n-sided polygon, $A_n = P_n \cdot r/2$ where A_n is the area of the polygon, P_n is its perimeter, and r is the radius of the inscribed circle. The area of the polygon is the sum of the areas of the isosceles triangles of which the polygon is composed. The base of such a triangle is proportional to its height (see **fig. 9**), which is the radius of the circle inscribed in the polygon. Thus, the perimeter of the polygon is also proportional to the radius r of the inscribed circle (see **fig. 9**).

For each one of the triangles—

- the side of the triangle is $a = 2r\tan\alpha$ where $\alpha = \pi/n$; and
- the area of the triangle is $A_\Delta = ar/2 = r^2\tan\alpha$.

For a regular n-gon, $P_n(r) = na = n(2r\tan\alpha) = 2nr\tan\alpha$. Then $A_n(r) = n \cdot r^2 \tan\alpha$; and $A'(r) = 2nr\tan\alpha$.

A regular polygon fulfills the derivative relationship between the area A and the perimeter P: $A'_n(r) = P_n(r)$ where r is the radius of the inscribed circle.

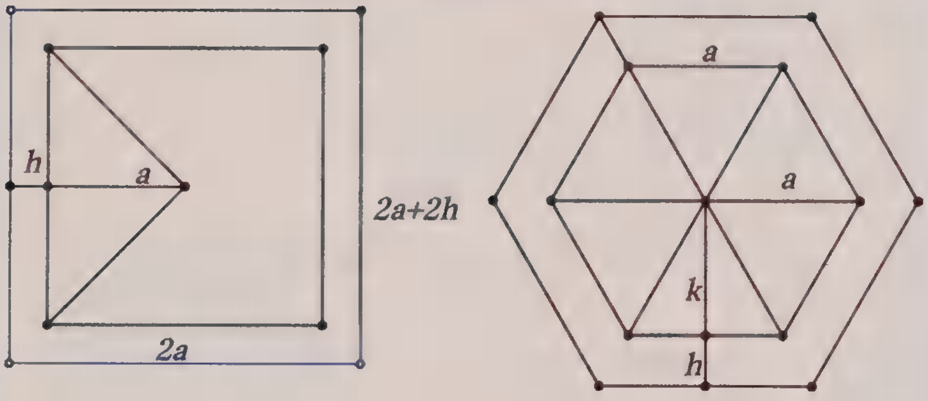
Note that $\tan\alpha$ and n are values specific to each polygon. Denoting $2n\tan\alpha$ as a constant C simplifies the expressions and highlights the derivative relationship:

$$P_n(r) = C \bullet r$$
$$A_n(r) = \frac{1}{2}C \bullet r^2 \rightarrow$$
$$A_n'(r) = P_n(r)$$

PLATONIC SOLIDS

Our next step is to move from regular polygons to regular solids—the five Platonic solids. (The analogy has already been established for a cube.)

As with the two-dimensional case, the volume of



$$A'(a) = P(a)$$

$$k = \frac{a\sqrt{3}}{2} \rightarrow a = \frac{2k}{\sqrt{3}} = \frac{2k\sqrt{3}}{3}$$

Fig. 7 The analogy between a square and a hexagon is attained.

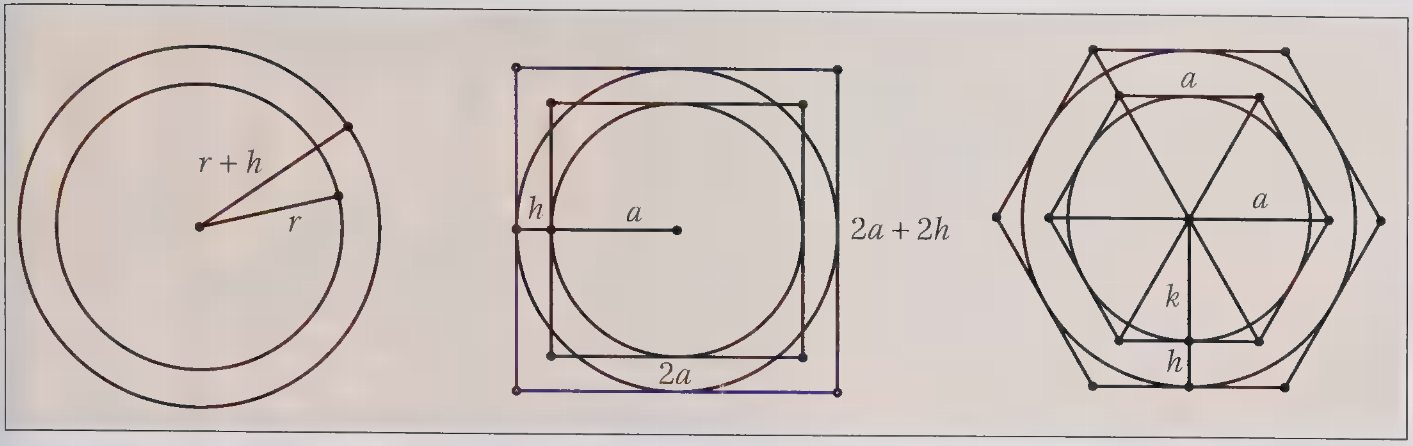


Fig. 8 The analogy holds for a circle, a square, and a hexagon.

a platonic solid with M faces can be considered as the sum of the volumes of M pyramids, where the base of a pyramid is the face of the solid and the height is the radius R of the sphere inscribed in the solid. (Each platonic solid circumscribes a sphere.) What follows is a proof that the derivative of the volume of a platonic solid equals its surface area when the volume and the surface area are functions of R. **Figure 10** uses an octahedron to exemplify the relationships between the radius R of the sphere inscribed in the solid and the radius r of the circle inscribed in a face of the solid.

Then $V(R) = MSR/3$ where R is the radius of inscribed sphere, S is the surface area of one face, and M is the number of faces. From the previous section (see **fig. 9**), $S = nr^2 \tan\alpha$ where r is the radius of the circle inscribed in a face of the solid and n is the number of sides of the face: Since $r = R\tan\beta \cdot \cos\alpha$, then

$$S = nR^2\tan^2\beta \cdot \cos^2\alpha \cdot \tan a$$
$$= \frac{nR^2\tan^2\beta\sin(2\alpha)}{2}.$$

Therefore, the surface area of the solid is

$$A(R) = \frac{MnR^2\tan^2\beta\sin(2\alpha)}{2},$$

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and

$$V(R)=\frac{MSR}{3}=\frac{1}{3}M\bullet\frac{1}{2}nR^2\tan^2 b\bullet\sin(2\alpha)\bullet R$$
$$=\frac{MnR^3\tan^2\beta\bullet\sin(2\alpha)}{6}.$$

Then

$$V'(R)=\frac{MnR^2\tan^2\beta\bullet\sin(2\alpha)}{2}\rightarrow$$
$$V'(R)=A(R).$$

A platonic solid thus fulfills the derivative relationship between the volume V and the surface area A where R is the radius of the sphere inscribed in the solid.

We note that the values α and β are constant for each platonic solid with M faces. Denoting

$$\frac{Mn\tan^2\beta\bullet\sin(2\alpha)}{2}$$

as a constant C simplifies the expressions and highlights the derivative relationship: If $A(R) = CR^2$ and $V(R) = CR^3/3$, then $V'(R) = A(R)$.

The relationships between the area and the perimeter of a polygon and between the volume and the surface area of a solid, which are immediately noticed with circles and spheres, exist for other regular polygons and solids when appropriate variables for the measures are chosen. The specific choices of variables and slight variations on the familiar formulas help students see this relationship clearly.

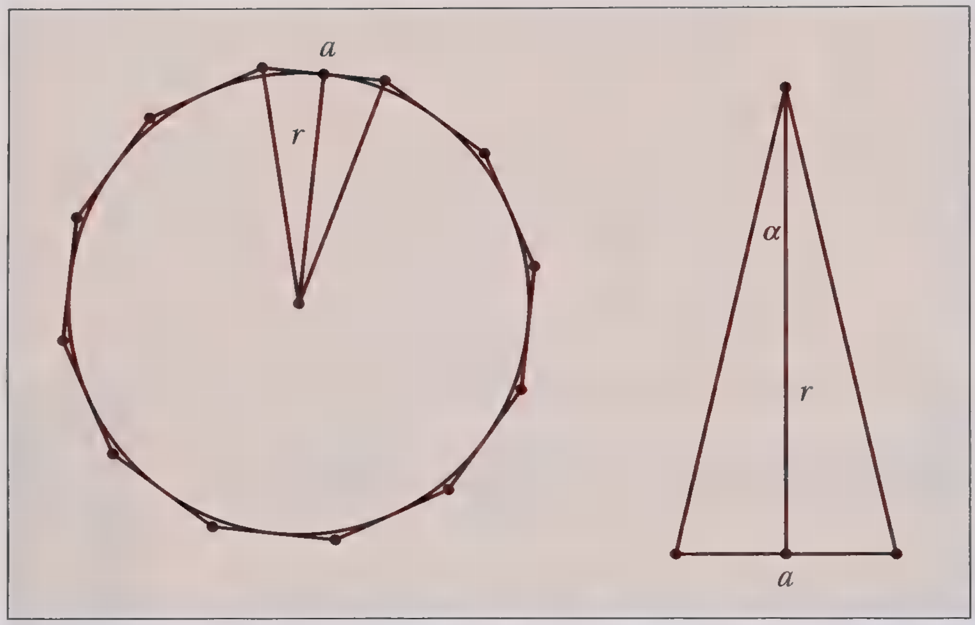

Fig. 9 The derivative of the area of a regular polygon equals its perimeter.

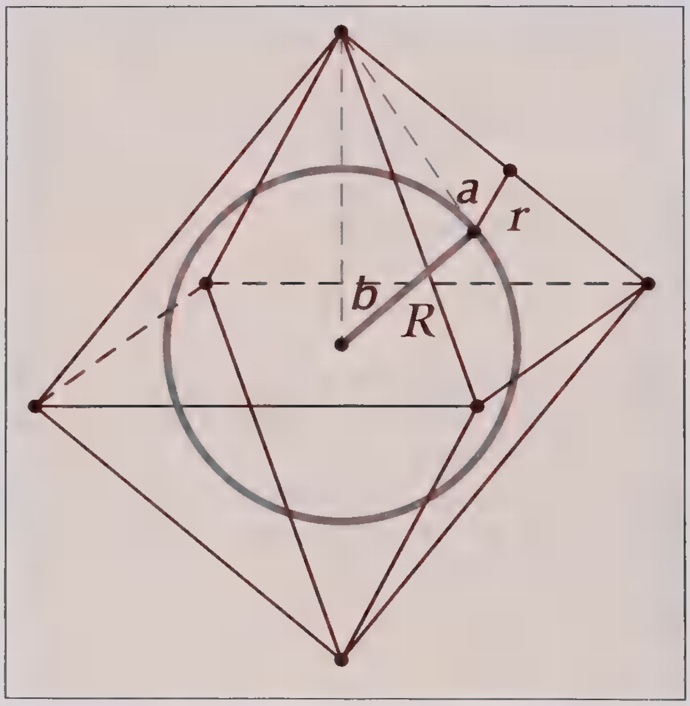

Fig. 10 The derivative of the volume of a platonic solid equals its surface area.

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MATHEMATICS teacher

ACTIVITIES for students

Flor Jacqueline Alarcón Mejía

Connecting Probability and Algebra

Algebra, probability, and sequences—all important curricular material—can be connected by a question that will challenge students: What is the probability P that the function $f(x) = x^2 + rx + s$ has real zeros when r and s are real numbers between 0 and 9, inclusive? This problem involves an infinite sample space, making it more interesting for students who have worked on probability problems with only finite sample spaces.

Students in ninth grade and up can accomplish this activity; however, it is probably best suited for a second-year algebra course. Students are expected to have basic skills using a TI-84 or similar calculator and should understand the difference between sequences and series. The activity can be presented as either an individual or a group activity but is more powerful when implemented in groups. When working in groups, students can review, clarify, and integrate concepts with other team members and can discuss possible solutions, communicate ideas, and share different points of view. I allot ninety minutes for the activity.

The activity allows students to deepen concepts such as calculating zeros of a quadratic function and calculating probabilities in an infinite sample space. Teachers can also introduce a step-by-step methodology that allows students to reflect on their findings as their comprehension increases.

Activities for Students appears six times each year in *Mathematics Teacher*, often providing in reproducible formats activity sheets that teachers can adapt for use in their own classroom. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more information on the department and guidelines for submitting a manuscript, please visit http://www.nctm.org/publications/content.aspx?id=10440#activities.

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ACTIVITY ANALYSIS AND SOLUTIONS

Case 1: r and s Are Integers

Case 1 generates a finite sample space. Students must consider the fact that r and s are integers.

First, students have to find fifteen random pairs (r, s), r and s being integers between 0 and 9, using the `rand` command of the calculator. Each pair of numbers (r, s) is used in the function $f(x) = x^2 + rx + s$. For example, the pair (5, 6) produces the function $f(x) = x^2 + 5x + 6$. For each of the fifteen random functions, students must then determine whether or not the function has real zeros. Students prefer to use the discriminant to determine whether the zeros of the equations are real. Because the process has to be done fifteen times, students may prefer to enter the calculator program shown in **figure 1** to evaluate the discriminant of the quadratic formula; they can then decide whether or not the function has any real zeros. On the basis of the results of their own table, the students must determine the relative frequency of the function $f(x) = x^2 + rx + s$ having real zeros.

```
PROGRAM: DISCRIM
Disp "x^2+Rx+S"
Prompt R, S
R^2-4S→Z
If Z ≥ 0
then
Disp "There are real zeros"
```

Fig. 1 Students could use this TI-84 program to determine whether the randomly generated function has real zeros.

This procedure allows only an approximation for an answer to the initial question; it is important for students to go through these steps because the process allows them to have a better understanding of what they are looking for. Students can see the elements that make up the sample space and how, depending on the number of elements taken into account, the value of the relative frequency is modified. Finally, the process encourages them to seek new strategies to solve the problem. After students finish the manual process, we suggest a strategy in which they will construct a diagram to represent the conditions of the problem and then use it to find the probability that the function $f(x) = x^2 + rx + s$ has real zeros.

For the diagram, students must take into consideration the following system of inequalities: $0 \le r \le 9$; $0 \le s \le 9$; and $r^2 - 4s \ge 0$. The first two inequalities are inferred because r and s are integers between 0 and 9, and the third inequality is the condition for the discriminant of the equation $x^2 + rx + s = 0$ to be positive so that there can exist real solutions. We expect that students will notice this fact to be able to solve problem 4.

In **figure 2**, each dot represents a pair (r, s) that determines a function of the form $f(x) = x^2 + rx + s$, and each of the dots marked with an X represents a pair (r, s) that determines a function of the form $f(x) = x^2 + rx + s$ with real zeros. To find the probability that the function $f(x) = x^2 + rx + s$ has real zeros, students must count the total number of

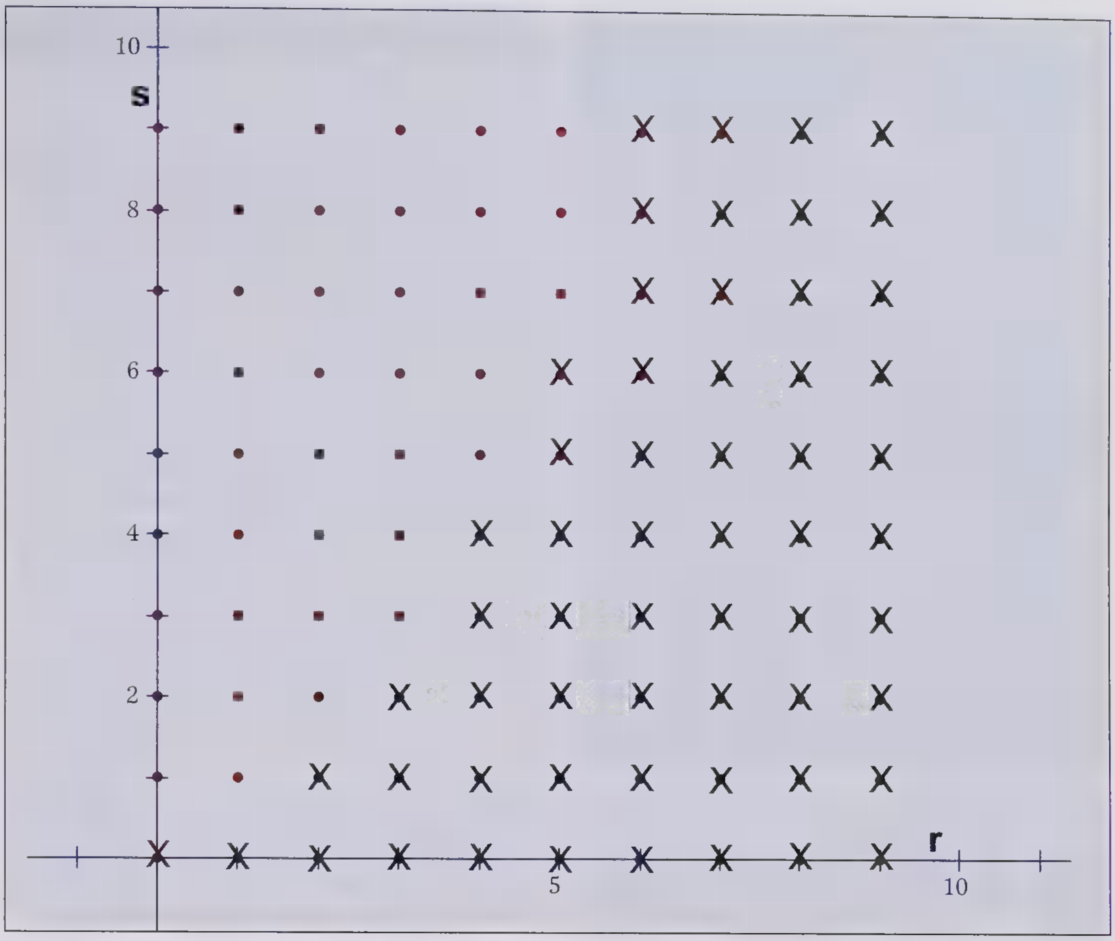

Fig. 2 Successful outcomes are represented by the Xs in the diagram.

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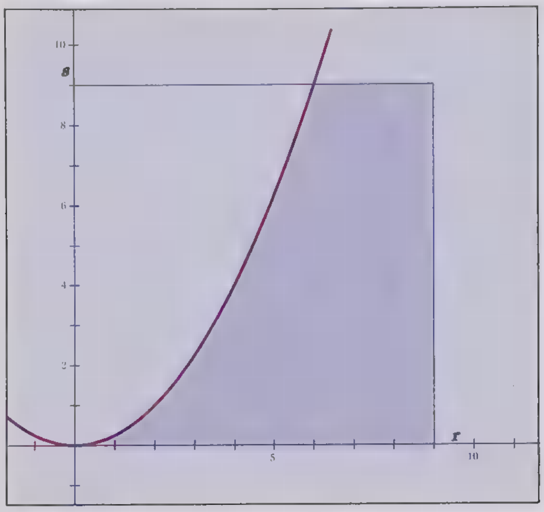

Fig. 3 The shaded area represents the successful outcomes.

dots marked with an X and then divide that number by the total number of dots (100): 59/100 = .59.

Many students will be astonished by how simple and easy it is to understand this graphic solution. They can count the elements in the sample space and see the result in a tangible, concrete way.

Case 2: r and s Are Real Numbers

Now students are to determine the probability that the function $f(x) = x^2 + rx + s$ has real zeros when r and s are real numbers between 0 and 9, inclusive. Students must be aware that this sample space is different. The new sample space, determined by the inequalities $0 \le r \le 9$ and $0 \le s \le 9$, has infinitely many points contained in the square with vertexes (0, 0), (9, 0), (9, 9), and (0, 9). The shaded area in **figure 3** shows the pairs (r, s) that represent real solutions for $f(x) = x^2 + rx + s = 0$ as determined by the inequality $r^2 - 4s \ge 0$.

Because the numbers in the pairs of the solution region are real, the number of pairs that satisfy the fact that the function $f(x) = x^2 + rx + s$ has real zeros cannot be counted, unlike in case 1, where r and s are integers. So it is necessary to find the areas of the shaded section of the figure and the area of the square and then calculate their ratio.

To find the area of the shaded section, divide it into two regions, as shown in **figure 4**, and then find the area of each shaded section. The area of the rectangle in region II is 27 (width 3 and length 9). To find the area of region I, we subdivide it into rectangles, as shown in **figure 5**. The idea is to use the areas of these smaller rectangles to approximate the area of region I. Here, it is possible to introduce the method of exhaustion as a way to find the area of a shape by inscribing a sequence of rectangles whose areas converge on the area we seek.

Table 1 shows the area of each rectangle $R1$ to $R6$. To obtain an approximation of the area from region I, sum the six areas:

Approximate area of region I

$$= \frac{1}{4} + 1 + \frac{9}{4} + 4 + \frac{25}{4} + 9 = 22.75$$

A best approximation of the area from region I can be made with a calculator using decreasing values each time for the width of the rectangles. **Table 2** shows the approximation of the area of region I according to three different values for the width of rectangles. From the data in the table, we can infer that the area of region I is approximately 18.

According to these results, the probability that the function of the form $f(x) = x^2 + rx + s$, with real numbers r and s, has real zeros is found as follows:

$$P = \frac{\text{Area region I} + \text{Area region II}}{\text{Area of square } ABCD}$$

$$\approx \frac{18 + 27}{81} \approx \frac{45}{81} \approx \frac{5}{9}$$

This calculation is based on the following principle of probability: If a point P of region S is picked at random, then the probability that P is in region R is $\text{Area}_R/\text{Area}_S$.

After working the problem using first a finite sample space and then an infinite sample space, students will reach the conclusion that the probability that the function $f(x) = x^2 + rx + s$ has real zeros when r and s are real numbers between 0 and 9, inclusive, is close to 5/9.

Problems 15 and 16 are intended to have students reflect on the difficulties they had while doing this classwork. Here, it is most important for students to identify the concepts they were able

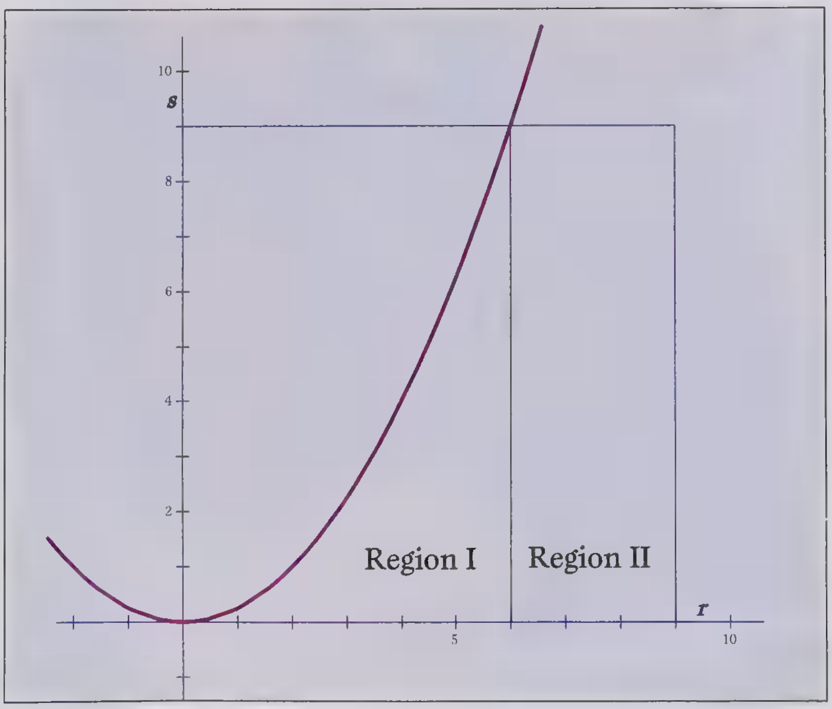

Fig. 4 Dividing the shaded region into a rectangle and a region bounded by a curve helps determine the area.

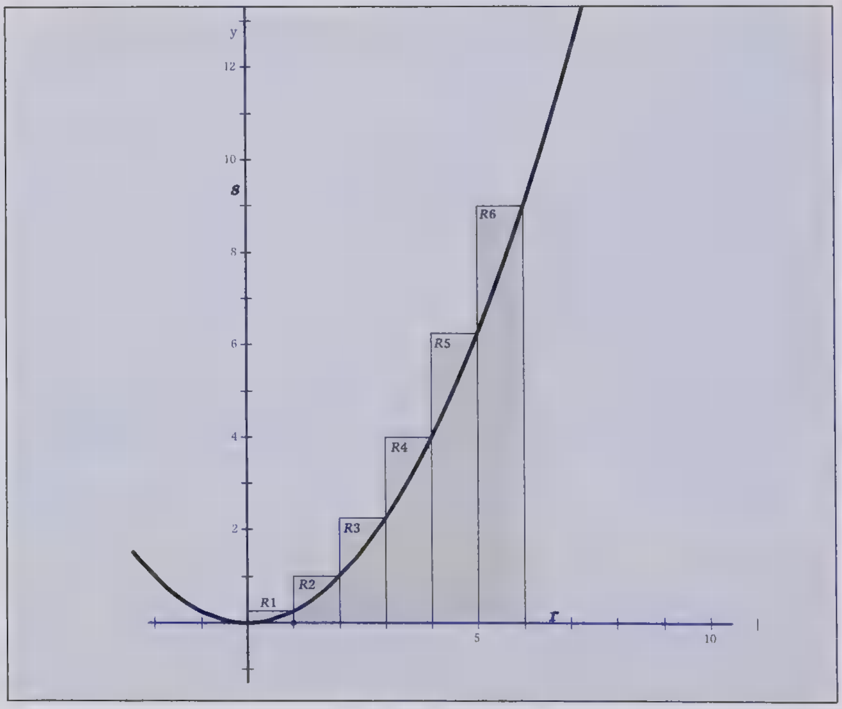

Fig. 5 The method of exhaustion can be used to determine the area of region I.

Table 1 Areas with $n = 6$

Rectangle 1 ($R1$)	Area $= 1\cdot\frac{1}{4}=\frac{1}{4}$
Rectangle 2 ($R2$)	Area $= 1\cdot 1 = 1$
Rectangle 3 ($R3$)	Area $= 1\cdot\frac{9}{4}=\frac{9}{4}$
Rectangle 4 ($R4$)	Area $= 1\cdot 4 = 4$
Rectangle 5 ($R5$)	Area $= 1\cdot\frac{25}{4}=\frac{25}{4}$
Rectangle 6 ($R6$)	Area $= 1\cdot 9 = 9$

Table 2 Better Approximations of the Area of Region I

Width of the Rectangles	Approximate Area of Region I
0.2	18.91
0.1	18.4525
0.01	18.045025

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to clarify and also to be aware of the importance of following a step-by-step procedure to answer a question.

WHAT STUDENTS LEARN

This activity allowed students to review some probability concepts and integrate them with some algebra concepts. By working in small groups, students learned to communicate, share their questions and doubts, and analyze group members' proposals. In addition, students learned to use a graphical approach in the resolution of probability problems involving an infinite sample space.

The two cases developed in the activity and the fact that they passed from discrete to continuous allowed students to have a better understanding of the problem and to find a correct solution. Determining the areas for the case in which the sample space is infinite can motivate students to consider future courses in calculus.

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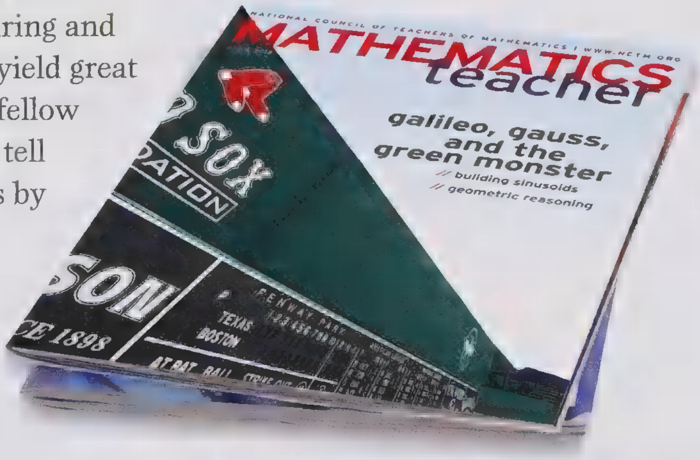


ACTIVITY: Quadratic Equations and Probability

Objective: To find the probability that the function $f(x) = x^2 + rx + s$ has real zeros when r and s are real numbers between 0 and 9, inclusive.

Case 1: r and s Are Integers

1. Use your calculator to generate fifteen random number pairs (r, s) of integers between 0 and 9, inclusive. Use these commands: **MATH** → **PRB** → **randInt(0, 9, 2)** → **ENTER**; repeat until you have obtained fifteen pairs. Enter the pairs in the table below.

Pair	r	s	**Function** $f(x) = x^2 + rx + s$	**Does** f **have real zeros? (yes or no)**
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				

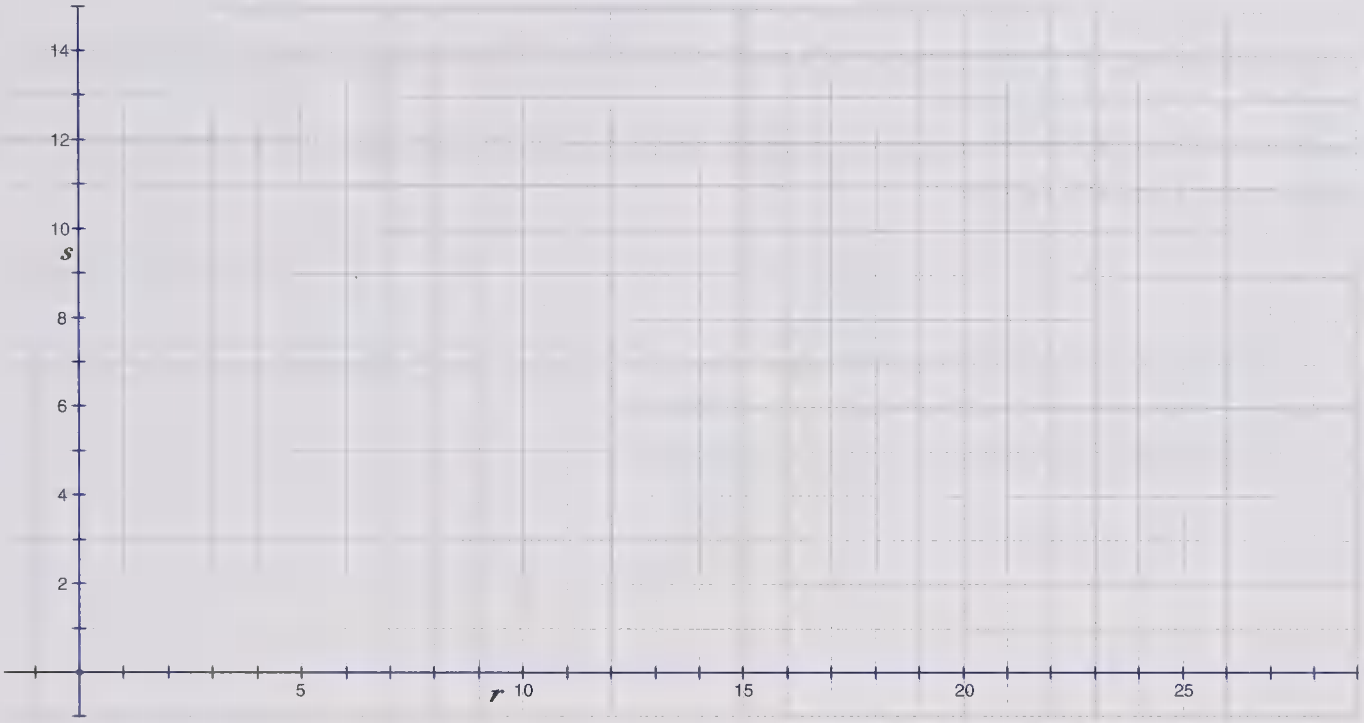
2. Use your results from the table to find the relative frequency (ratio of number of successful outcomes to the total number of outcomes) that the function $f(x) = x^2 + rx + s$ has real zeros:

$$\text{Relative frequency} = \frac{\text{Number of successful trials}}{\text{Total number of trials}} =$$

From the May 2013 issue of MATHEMATICS teacher

ACTIVITY: Quadratic Equations and Probability (continued)

3. If r and s are integers between 0 and 9, inclusive, place a dot to indicate all pairs (r, s) in the sample space.



4. What must be true for integers r and s to make the function $f(x) = x^2 + rx + s$ have real zeros?

5. Using the diagram shown in problem 3, mark with an X each point that satisfies the condition found in problem 4.

6. Use the graphic representation from problem 5 to find the probability that the function $f(x) = x^2 + rx + s$ has real zeros:

$$P = \frac{\text{Total number of dots marked with an X}}{\text{Total number of dots}} =$$

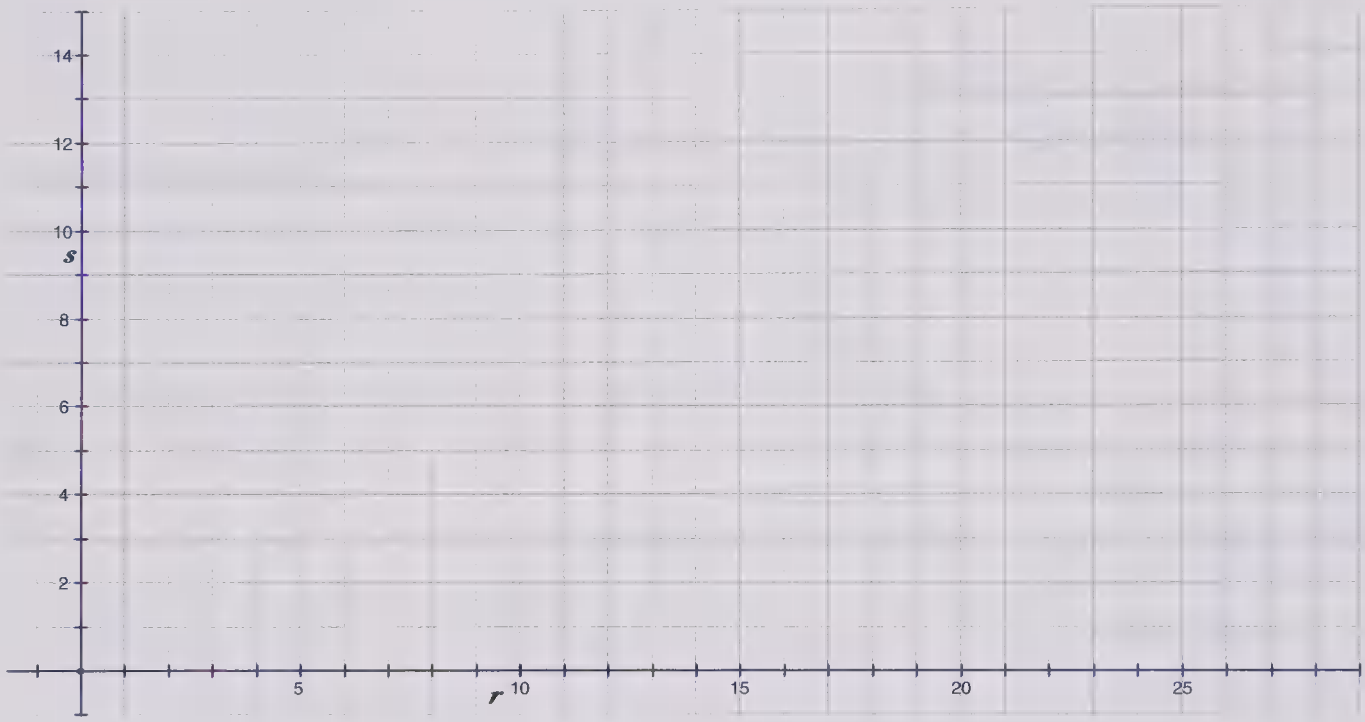
ACTIVITY: Quadratic Equations and Probability (continued)

Case 2: r and s Are Real Numbers

1. If r and s are real numbers between 0 and 9, inclusive, how many elements are there in the sample space?

   How is this sample space different from the sample space you found when r and s were integers?

2. In the coordinate plane below, indicate the sample space formed by all pairs (r, s) where r and s are real numbers between 0 and 9, inclusive



3. What must be true for real numbers r and s to make the function $f(x) = x^2 + rx + s$ have real zeros?

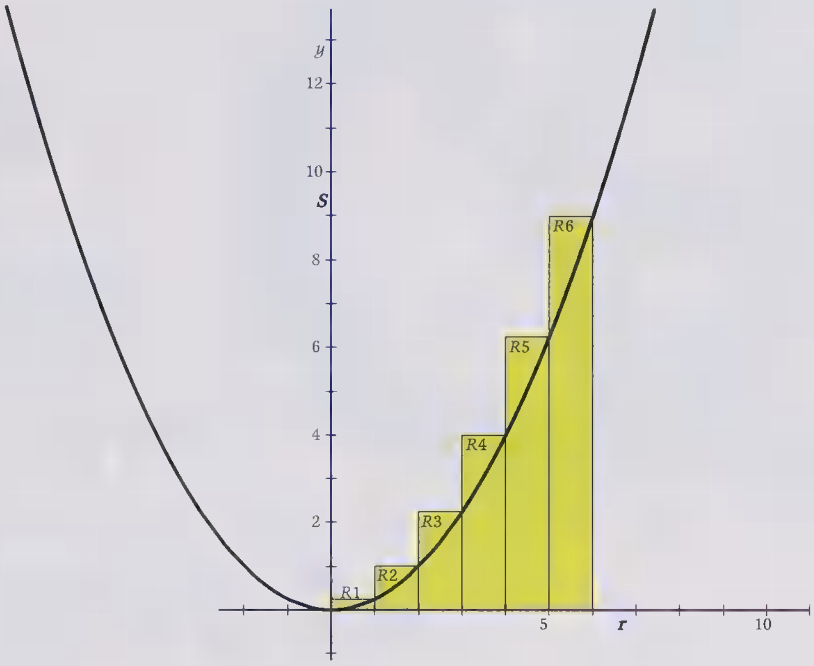
4. In the rs-coordinate plane shown in problem 2, shade in the region that satisfies the conditions stated in problem 3.

5. Split your shaded figure into exactly two regions in which one region is the largest rectangle possible. Label the rectangle region II and the other area region I.

6. Calculate the area of the rectangle (region II):

   Area rectangle =

7. Divide the remaining figure (region I) into rectangles as shown below.



8. Calculate the area of the rectangles that you constructed in problem 7 and complete the table below.

Rectangle 1 ($R1$)	Area $= 1 \bullet \frac{1}{4} = \frac{1}{4}$
Rectangle 2 ($R2$)	Area =
Rectangle 3 ($R3$)	Area =
Rectangle 4 ($R4$)	Area =
Rectangle 5 ($R5$)	Area =
Rectangle 6 ($R6$)	Area =

9. An approximate area of region I is the sum of the areas of the rectangles. The sum is ________________________.

10. What can you do to make the approximation of the area of region I more accurate?

11. You can refine this result by systematically reducing the width of the rectangles. To use your TI-84 calculator to do this, enter `x²/4` in `Y₁`. From the home screen, use the `LIST` commands to obtain `sum(seq(w*Y₁,x,0,6,w)`. Here, w is the width of each of the six small rectangles. If we substitute 1 for w, the result will be the same as that obtained in problem 9. Verify this result.

12. You can change the value of w to make the area of region I more accurate. Complete the following table:

Width of Rectangles (w)	**Sum of the Areas**
0.2	
0.1	
0.01	

13. Analyze your table. What can you say about the area of region I as you reduce the width of the rectangles?

14. Using your results, find the probability that the function $f(x) = x^2 + rx + s$ has real zeros when r and s are real numbers between 0 and 9, inclusive.

$$P = \frac{\text{Area Part I} + \text{Area Part II}}{\text{Area of square}} =$$

15. Did you have any difficulties in solving the problem? If so, explain.

16. What have you learned by doing this activity?

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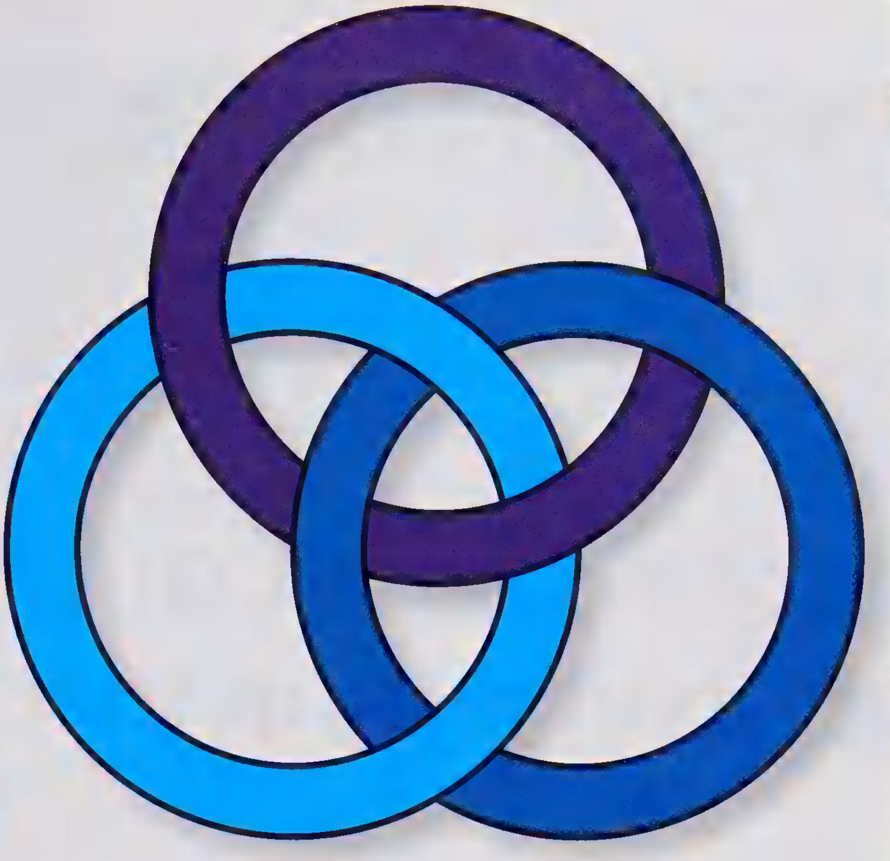

2015 Focus Issue

Creating Classroom Communities

NCTM guides mathematics teachers toward equitable teaching by emphasizing the importance of classroom communities. The Editorial Panel of *Mathematics Teacher* invites teachers, teacher educators, and education researchers to share their experiences in building classroom communities. We encourage submissions that will help *MT* readers learn new ways to capitalize on the strengths that cultural, racial, and linguistic diversity bring to our classrooms and schools.

Classroom communities embrace individuals.

- How can we foster a sense of belonging in our classrooms?
- How can we learn about our students—their interests, issues that might be important to them, their languages, and their racial or ethnic communities?
- How can we incorporate this knowledge in our lessons and assessments? What are examples of effective tasks that highlight strengths of individual students? How can we balance individual, cooperative, and whole-class activities?

Classroom communities foster communication.

- How do we organize lessons so that students can share their mathematical ideas or solutions?
- What classroom norms are effective in facilitating communication?
- How can we encourage students to listen to, critique, and build on other students' mathematical thinking?
- How can we communicate high standards for students?
- How can technologies foster student communication about mathematics?
- What strategies are successful in removing barriers to student participation and engagement in mathematics?

Please submit manuscripts at **mt.msubmit.net** by **May 1, 2014**. Be sure to enter the call's title (Creating Classroom Communities) in the Department/Calls field. No author identification should appear in the text of the manuscript. See **www.nctm.org/publications/content.aspx?id=22602** for manuscript guidelines. If you have ideas related to this topic and would like to discuss them before sending a manuscript, please contact Albert Goetz, agoetz@nctm.org.

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CONNECTING RESEARCH to teaching

Heather Lynn Johnson

Reasoning about Quantities That Change Together

REASONING ABOUT COVARYING QUANTITIES

Imagine a soda bottle being filled at a constant rate. How might the volume of soda in the bottle change with respect to the height of soda in the bottle?

To respond, we need to consider how the quantities of volume and height *covary*, or change together, in relation to the shape of the bottle. That is, we need to engage in quantitative reasoning that involves describing rates of change. Although we usually associate thinking about rates of change—particularly, varying rates of change—with calculus, the foundation of these ideas begins much earlier. In this article, I share data from interviews with secondary school mathematics students to illustrate two different types of reasoning that can be observed in such situations.

The Common Core State Standards Initiative (CCSSI 2010) identified reasoning quantitatively as an essential mathematical practice, stating that students should "make sense of quantities and their relationships in problem situations" (p. 6). This is necessarily more than just writing a label next to a number. To reason quantitatively, students should be able to identify and think about attributes that they could measure (Smith and Thompson 2008). For example, in the bottle-filling task, students could measure the height of the level of soda in a bottle at a particular moment in time. However, determining several data values for height versus time may not be required for students to make sense of each quantity and how it varies.

One way in which teachers can support students' quantitative reasoning is to engage them in tasks that involve quantities that change together. The bottle-filling task outlined here could be used with students at a variety of grade levels. Take a moment to think about how a student who has not taken calculus might respond to the task.

THE BOTTLE-FILLING TASK

The graph shown in **figure 1** represents a relationship between the volume and the height of liquid in a bottle being filled at a constant rate. Students should consider these two questions:

1. How is the volume of the liquid in the bottle changing as the height of the liquid in the bottle increases?
2. Sketch a bottle that the graph could represent.

Note the choice to present volume as a function of height rather than height as a function of volume. Carlson, Larsen, and Lesh (2003) found that university students working on a bottle-filling

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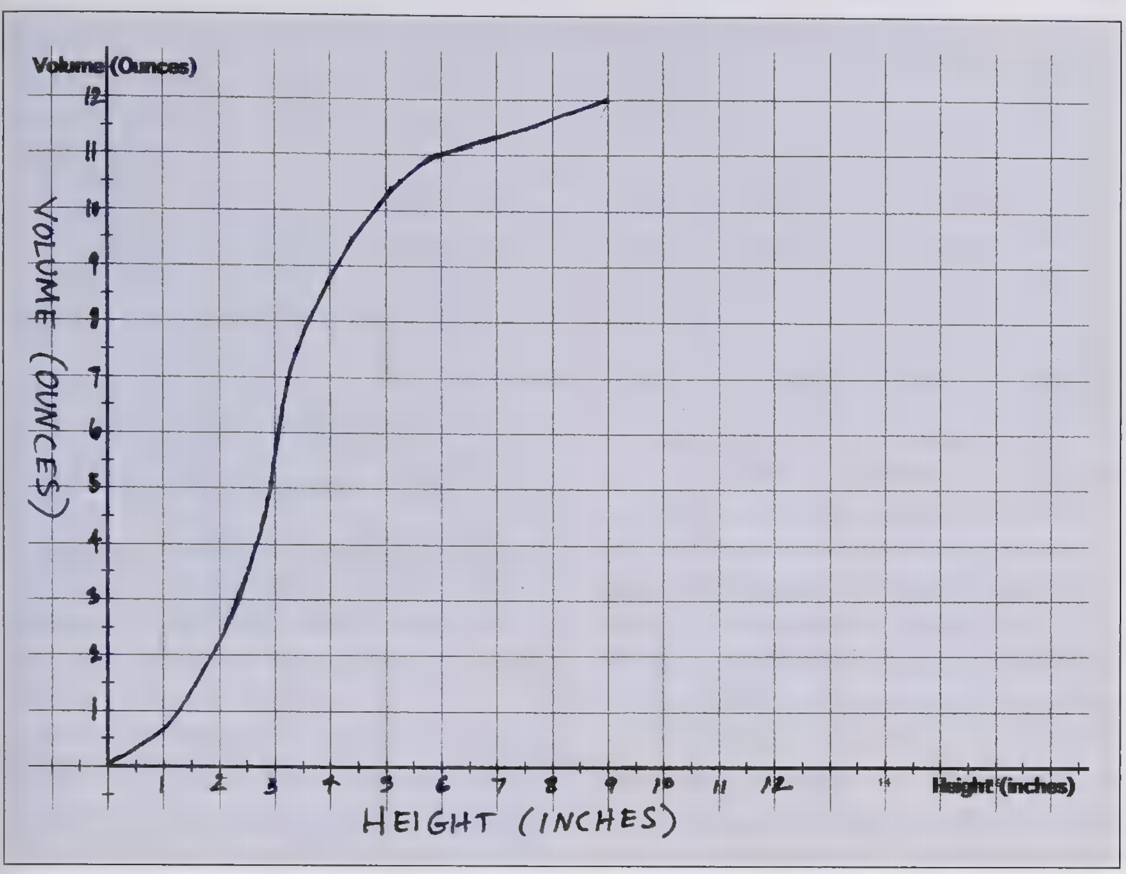

Fig. 1 The graph represents a relationship between the volume and the height in a bottle being filled at a constant rate.

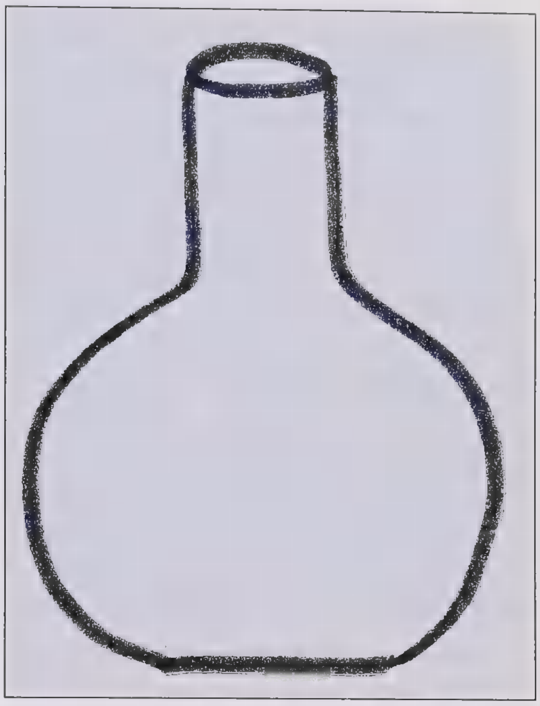

Fig. 2 This bottle corresponds to the graph in the bottle-filling task.

problem that represented height as a function of volume treated volume as if it were time. Imagine watching the increase in the height of the liquid in a filling bottle with nonconstant width, such as the one shown in **figure 2**. Because the height of the liquid would appear to increase more slowly in wider sections of the bottle and more quickly in narrower section, I thought it might be less likely for high school students to treat height as if it were time. Thus, representing volume as a function of height could reveal more of students' reasoning about the two quantities.

Students could respond to this task in a variety of ways. Clearly, volume increases as height increases. To determine the shape of the bottle, students need to attend to how the increase in volume varies in relation to height. Because the graph incorporates numerical values, students might focus on particular amounts of change by determining corresponding values of height and volume and considering the differences in both height and volume. To propose a shape for the bottle, students need to coordinate their observations about the changes in volume in relation to changes in height. The sketch of a bottle that corresponds to the graph is shown in **figure 2**.

SETTING UP THE TASK

When teaching AP Calculus, I wondered how students who had not yet taken calculus might respond to questions about varying rates of change. Six high school students took part in five individual interview sessions during which they worked on one or more mathematical tasks and explained their thinking. They worked on the bottle-filling task during the fifth interview session.

Before presenting the task, I asked each student to talk about what volume meant without making any calculations. I did so expecting to learn more about how students made sense of volume as something that they could measure. Each student described volume as the amount of space being taken up by an object. Their descriptions let me know that they could reason about volume as more than just the result of a calculation. By closely examining the reasoning of individual students, teachers can anticipate how students might make sense of changing quantities when working on such tasks.

I share three students' work on the bottle-filling task. Students responded to the first question orally. For the second question, students sketched a bottle and then explained how their sketches related to the graph. I include direct quotes and sketches from Jacob, Mason, and Hannah (all names are pseudonyms). At the time of the study, Jacob and Mason were enrolled in a precalculus course, and Hannah was enrolled in a geometry course.

RESPONSES TO THE TASK: JACOB AND MASON

Both Jacob and Mason described relationships between volume and height by comparing amounts of change in each quantity.

Jacob's Response

> Well, . . . here there's more volume [*points to the volume axis of the graph given in the task*]. I'll just use the point (2, 2). See, like, it's more volume than height [*points to the height axis*]. So there's more . . . if it was (2, 2), then it would have been constant. So there's more volume than height, so that means it's, like, fat at the bottom, and then here [*refers to the portion of the graph near the point* (*3, 5.5*)], it starts getting skinnier and skinnier and skinnier.
>
> This curve [*refers to the concave-up portion of the graph*] means that the bottle is, like, . . . it's not getting very high for how much soda's filling up. So that means it's like this [*draws the bottom portion of the bottle shown in* **fig. 3**].

> And you're not get- [*brief pause; moves marker along the concave-down part of the sketch shown in* **fig. 3**], but up here . . . it's getting skinnier, so that way you're getting more height with less soda. So the bottle looks like that, kind of [*draws the upper portion of the bottle shown in* **fig. 3**].

Mason's Response

> I'd say that it's filling up pretty fast here [*moves marker along the curve in the given graph, beginning at the origin*]. And it hits a certain point in that specific bottle where it starts to slow down a little bit until it reaches its height, its final height.
>
> It is taking up more volume than the height here. I got this backwards. It's taking up more of volume. Yeah, I have it backwards [*scratches out bottle shape on left; see* **fig. 4**]. It's going to be wider down at the bottom. It gets smaller up at the top because it's taking up more inches, and it's less amount of volume getting into it. So I think it is going to start out wide. And then once it reaches that certain point, it's just going to taper off.

Analyzing These Responses

Jacob compared amounts of increase in volume and height and made connections to the shape of a bottle. Using point (2, 2) as an example, he explained how to tell whether the bottle would be wide or narrow in that section. Because the graph of volume versus height was to the left of point (2, 2), Jacob determined that there would be "more volume than height," meaning that the bottle would be wide in that section.

Jacob sketched the bottle in two parts—wider at the base and narrowing as it reached the top. He made connections between the bottle and the graph, as shown by the arrows in **figure 3**. To determine when the bottle would be wide or narrow, he compared amounts of volume and height. When volume was increasing more (e.g., "it's not getting very high for how much soda's filling up"), Jacob drew a wide part in the bottle. When height was increasing more (e.g., "getting more height with less soda"), he drew a narrow part in the bottle.

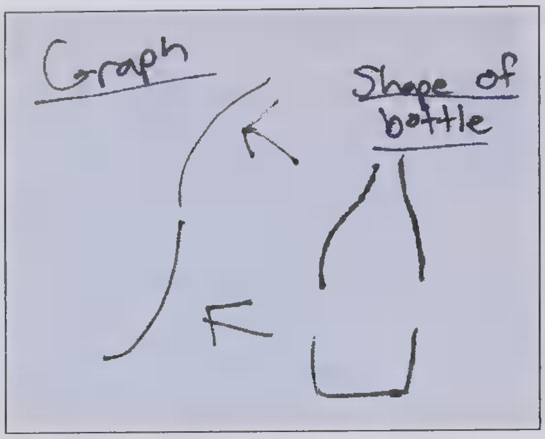

Fig. 3 Jacob sketched this bottle.

Mason identified sections when the bottle was filling differently ("filling up pretty fast here," "starts to slow down a little bit"). Even though he described different rates of filling, it is difficult to tell how he might have been relating the quantities of volume and height. If students provide a partial response to one part of a task, their responses to other parts might give more information about their reasoning. Mason's second explanation demonstrates comparison of amounts of volume and height.

Mason made two attempts to sketch a bottle. He first drew a bottle that had multiple inward and outward curves (see **fig. 4**, left). When using the graph to explain why the bottle would look that way, he scratched out the first sketch. The second bottle that Mason drew (see **fig. 4**, right) was much closer to the anticipated solution (see **fig. 2**). When drawing the second bottle, he explained that the change in the bottle shape resulted from paying attention both to "the numbers" and to the shape of curve in the given graph. Moving a finger over the height and volume axes, he went on to say that "looking at the numbers" meant paying attention to amounts of change in height and volume in particular sections of the bottle. By making comparisons between amounts of volume and height, Mason was able to recognize inconsistencies in the shape of the first bottle sketched and draw a more appropriately shaped bottle.

Both Jacob and Mason compared amounts of change in volume to amounts of change in height in different sections of the bottle and used their comparisons to make predictions about where the bottle would be wider or narrower. Even though the sketches that they drew are not completely accurate, each sketch shows a bottle narrowing and nearly cylindrical toward the top.

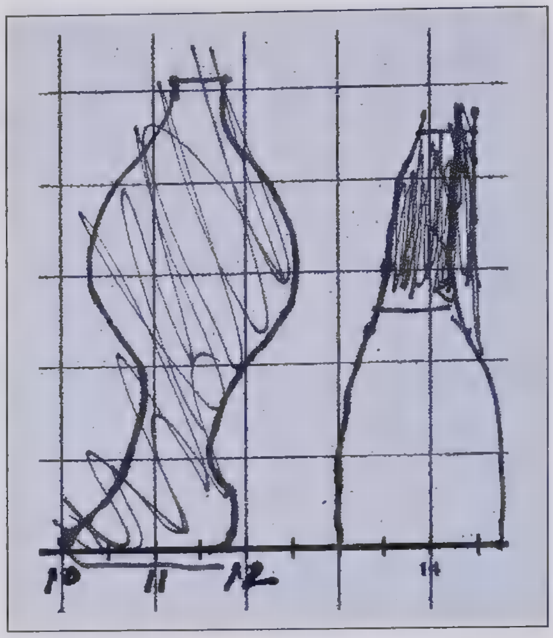

Fig. 4 This bottle was drawn by Mason.

RESPONSE TO THE TASK: HANNAH

Hannah described variation in how the volume changed as the height increased.

Hannah's Response

> Towards the bottom of it, like, it doesn't increase as much, but as you go along, it definitely increases more. The volume and the height . . . it increases more as you go. The volume of it increases a lot, like, from here to here [*puts two fingers on the graph, one at the point where the height is approximately 3 inches and the other where the height is approximately 4 inches*], like, compared to down here [*points near the origin*]. So there would be a bigger volume for that. Towards the middle it starts … it gets smaller as you go to the cap of the bottle.
>
> It would start off smaller because the increase of volume starts off slower, and then it would start to get larger, so it would have this kind of curve to it [*draws lower-left side of the bottle; see* **fig. 5**]. And then, … as soon as you get up here into this, here the bottle would get smaller again, as it went to this cap [*continues to draw left side of the bottle*]. So it would be sort of like a vase shape . . . Then [*finishes drawing the bottle*] it would just be sort of shaped like that because the volume down here, like, it

> shows on this graph, like, the volume starts to increase faster as it gets in towards around here. And then as you go up further and then as the height increases, the volume starts to slow, like slower increasing.

Hannah made distinctions between different kinds of increases in volume. For example, she said that at first the volume did not "increase as much" and then the volume "definitely increases more." Hannah treated increases in volume as occurring in relation to a continually increasing height. It was as if she were able to imagine change happening in progress, linking the changing volume and the changing height.

When sketching the bottle, Hannah made connections between variation in the increase in volume and the curved shape of the bottle. Specifically, she used the variation in the increase in the volume to explain why the bottle would be shaped a certain way. Drawing the lower-left portion of the bottle that curved outward to the widest point (see **fig. 5**), she explained that the bottle would have this shape because the increases in volume would "start to get larger." Unlike Jacob's and Mason's sketches, Hannah's bottle shows considerable widening and narrowing before the narrower, more cylindrical portion at the top of the bottle.

ANALYZING ALL STUDENTS' REASONING

When working on the bottle-filling task, all three students described relationships between the changing quantities of volume and height. Jacob and Mason compared amounts of height and volume to determine the shape of a bottle that corresponds to the graph given in the task. In contrast, Hannah described variation in increases in volume in relation to increases in height (e.g., "slower" and "faster" increases). Further, she drew a bottle closest to the shape expected for this task.

By analyzing how students described relationships between the changing quantities, I identified two distinct forms of reasoning: simultaneous-independent (Johnson 2012a) and change-dependent (Johnson 2012b). Distinctions such as these are helpful for teachers as they analyze how students are making sense of situations involving quantities that change together.

Simultaneous-Independent Reasoning

Simultaneous-independent reasoning involves the activity of comparing. When interpreting a graph involving changing quantities, students who demonstrate simultaneous-independent reasoning use intervals (or sections) to compare amounts of change. The comparisons that students make could be between the magnitudes of the amounts. In this case, the quantity with the greater (or lesser) amount of change would be increasing more (or less) in that interval.

Students make these comparisons as if each quantity were changing independently of the other in relation to time. By making comparisons between amounts of change in volume and height, Jacob and Mason appeared to be using simultaneous-independent reasoning.

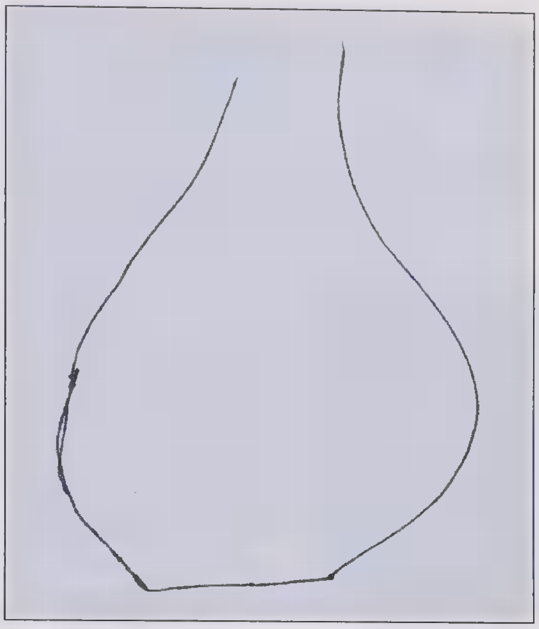

Fig. 5 Hannah sketched this bottle.

Change-Dependent Reasoning

Change-dependent reasoning involves the activity of coordinating. When interpreting a graph involving changing quantities, students who demonstrate change-dependent reasoning coordinate change in one quantity with change in another quantity. This process could involve imagining a point on a graph moving from left to right in conjunction with a point on the horizontal axis moving from left to right. As the quantity

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represented on the horizontal axis increases, the quantity represented on the vertical axis might increase or decrease. Thus, a student could conceive of variation in amounts of increase or decrease.

Hannah appeared to be using change-dependent reasoning by describing variation in how the volume changed as the height increased. Unlike Jacob and Mason, who compared amounts of change in volume and height, she indicated different kinds of increase in volume (e.g., describing volume as "increase faster" and then as "slower increasing").

IMPLICATIONS FOR TEACHING

When students reason about quantities that change together, the way in which they relate the changes in the quantities matters. Although simultaneous-independent reasoning is useful, it does have limitations. Simultaneous-independent reasoning involves making comparisons between static amounts of change in quantities, determining in each interval when one quantity will be increasing by a greater, lesser, or the same amount as another quantity. In contrast, change-dependent reasoning involves variation in how one quantity changes in relation to another quantity. Change-dependent reasoning draws more fully on the dynamic aspect of the changing quantities.

Research shows that students can treat graphs as pictures of physical objects (Leinhardt, Zaslavksy and Stein 1990). Consider how the left contour of the first bottle that Mason drew (see **fig. 3**, left) resembles the shape of the curved portion of the graph given in the bottle-filling task. When comparing amounts of change in volume and height, he noticed an error and made a new sketch.

Teachers can support students' efforts to interpret graphs by fostering students' use of simultaneous-independent reasoning. Asking questions that encourage students to pay attention to quantities is one way to do this. Before interpreting graphs, students could reason about height and volume for differently shaped bottles by filling the bottles (perhaps in increments of 2 ounces of liquid) and collecting the height and volume data for each bottle. (For suggestions on such activities, see Gomez and Wolfson [2012] or McCoy et al. [2012].)

Further, teachers can strengthen students' efforts to make sense of variation in rates of change by fostering their use of change-dependent reasoning. Asking students to move a point along the x-axis in conjunction with another point that is moving along a graph of a function is one way to do this.

Reasoning about quantities that change together is essential for understanding calculus concepts such as the derivative (Zandieh 2000). However, it can be nurtured much earlier to support students' understanding of that difficult-to-learn concept—rate of change (Thompson 1994). Teachers can foster student reasoning by paying close attention to the ways in which students describe relationships between quantities. This process better prepares students to draw on their informal ways of reasoning when the time comes to formalize relationships between varying quantities in calculus.

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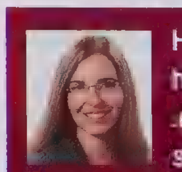

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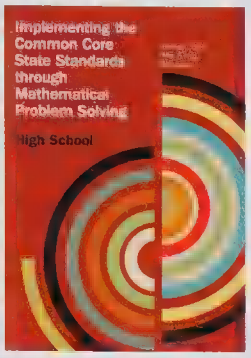
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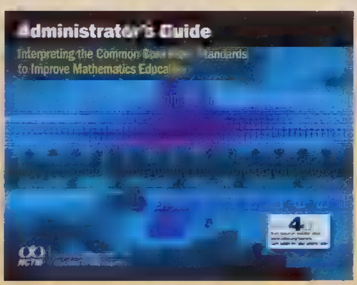
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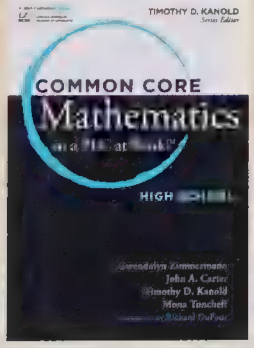

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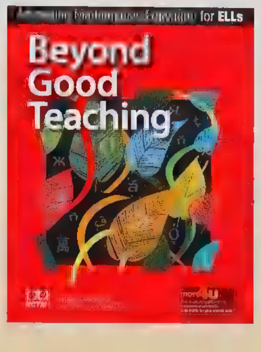

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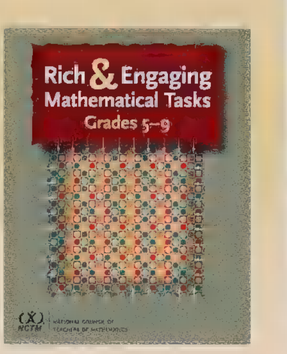

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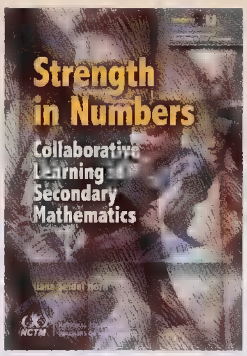

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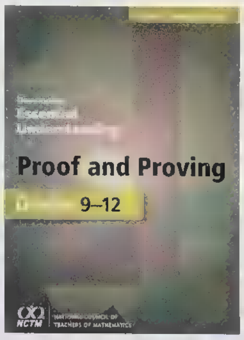

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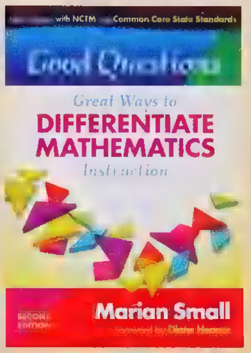

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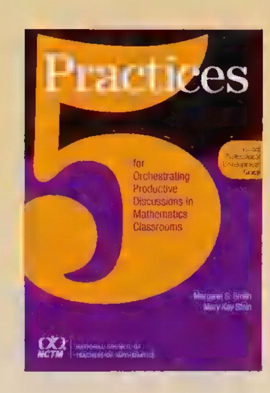

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Dean B. Priest, Ronald G. Smith, Christin Carlisle, and Rebecca Mays

The Diver Problem: The Surfer Problem in 3D

Mathematics is not a "handbag of tricks" but rather a discipline of inquiry and creativity, as Nabb (2010–11) notes, and he has shared his methods and excitement for the inquiry approach. By engaging calculus students in a search for examples of infinite series that meet certain conditions, or arguments that such series do not exist, Nabb appropriately aligns his teaching with the Standards for Mathematical Practice found in the Common Core State Standards (CCSSI 2010). In particular, three Standards for Mathematical Practice come to mind: (a) "make sense of problems and persevere in solving them"; (b) "reason abstractly and quantitatively"; and (c) "model with mathematics" (CCSSI 2010, pp. 6–8).

To enlarge the pool of mathematics teachers like Nabb, it is imperative that higher education faculty "also model the effective practices they believe teachers should employ" (NCTM 2000, p. 377). One problem that has provided many opportunities for us to accomplish this with our preservice teachers, using methods similar to Nabb's, is the surfer problem as presented by Copes and Kahan (2006). This problem is stated as follows:

> A surfer, shipwrecked on an island in the shape of an equilateral triangle, wants to build a hut so that the sum of its distances to the three beaches is minimal. Where should the hut be located?

The authors demonstrate several solutions to this problem, including a coordinate geometry proof and an area proof. In all cases, they show that the hut can be located anywhere on the island by proving that the sum of the distances equals the height of the triangle. In addition, they challenge readers to discover other approaches for themselves. In 2008, one of our students, Kena Gibson, had her original solutions to this surfer problem accepted for presentation at the regional Missouri/Arkansas/

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Subsequently, we were delighted to see Kalish's (2009) extensions to the surfer problem. Among other things, he shows that the hut can be located outside the triangle with the same results, assuming that we measure the distances to the extended lines of the equilateral triangle and permit signed distances. We immediately suggested to some of our students that they attempt to find their own original proofs of the Kalish results. In preparation for their senior capstone seminar, two of them collaborated to produce a proof of the extended surfer problem. Their proof, an analytic geometry proof, was easily applicable to other regular polygons, as also shown by Kalish. But this was not to be the end of their investigations.

With our prompting, the two students began to conjecture about whether results might be similar when working in three dimensions. Their musings were along these lines: "If we built a polyhedral structure underwater for deep-sea divers, where would we locate living quarters inside so that the sum of the distances from these quarters to the faces of the structure would be a minimum?" The first structure suggested was a right equilateral triangular prism, extending down into the ocean but open at the top for entrance into the structure. This idea was quickly discarded because, when modeled, it was essentially the two-dimensional surfer problem all over again.

Nevertheless, the students continued to exercise their mathematical skills of inquiry and creativity until one day one of them simply asked, "What would happen if we considered a tetrahedron?" In time, this question helped them develop conjectures that led to their first three-dimensional extension of the surfer problem. They quickly found two proofs by simply extending the two-dimensional analytical geometry and area proofs (as per Copes and Kahan) to three-dimensional analytical geometry and volume proofs. Here is their first extension.

THEOREM 1

Theorem 1 may be stated as follows:

> For a regular tetrahedron $PQRS$, the sum of the distances from any point *inside* the tetrahedron to its faces is a constant and equals the height of the tetrahedron.

Proof

An interior point, E, of the regular tetrahedron $PQRS$, when connected to the vertices, partitions the regular tetrahedron into four pyramids, all having one of the four congruent faces of the tetrahedron as a base (see **fig. 1**). The sum of the volumes of the four pyramids equals the volume of the regular tetrahedron. Therefore,

$$\frac{1}{3}A_B h_T = \frac{1}{3}A_1 h_1 + \frac{1}{3}A_2 h_2 + \frac{1}{3}A_3 h_3 + \frac{1}{3}A_4 h_4, \quad (1)$$

where A_B, A_1, A_2, A_3, and A_4 refer, respectively, to the areas of the bases and h_T, h_1, h_2, h_3 and h_4 refer to the heights of the regular tetrahedron and the four pyramids. Since $A_B = A_1 = A_2 = A_3 = A_4$, equation (1) implies that $h_T = h_1 + h_2 + h_3 + h_4$, thus proving theorem 1.

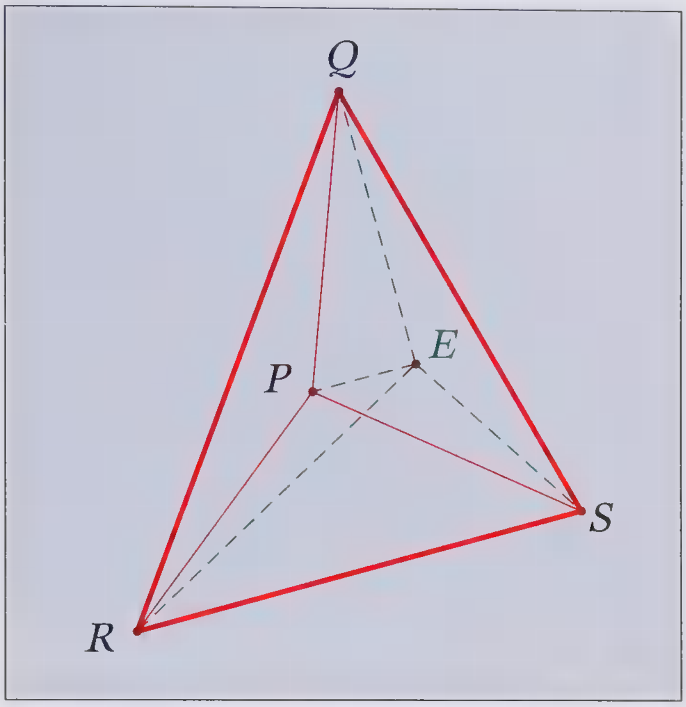


Fig. 1 Students can see four smaller tetrahedra when point *E* is in the interior of *PQRS*.

Before showing the students' three-dimensional extension involving points anywhere in space, we extend Kalish's definition of signed distances in relation to a line in two dimensions to signed distances in relation to a tetrahedron in three dimensions (Priest and Smith 2010). First, we note that the extended plane π of any face of the tetrahedron partitions three-space into three disjoint sets: the plane π and two half-spaces, one that contains the interior of the tetrahedron (H_1) and one that does not (H_2).

Definition 1

Let π denote the extension of one face of a regular tetrahedron $ABCD$. Letting M be any point in three-space and denoting the distance from M to π by $d(M, \pi)$, we define the signed distance from M to π, $SD(M, \pi)$, as $d(M, \pi)$ if M is in H_1 or in π and as $-d(M, \pi)$ if M is in H_2.

For purposes of computing the signed distance of a point to the extended faces of a tetrahedron, we observe the following. If π is a plane determined by two vectors $\overrightarrow{PA}$ and $\overrightarrow{PB}$ (see **fig. 2**), then $\overrightarrow{N} = \overrightarrow{PA} \times \overrightarrow{PB}$ is a vector normal to π. Also, if Q is any point in three-space and $t\overrightarrow{N}$ is the projection of $\overrightarrow{PQ}$ onto $\overrightarrow{N}$, then $(\overrightarrow{PQ} - t\overrightarrow{N}) \cdot \overrightarrow{N} = 0$. Hence,

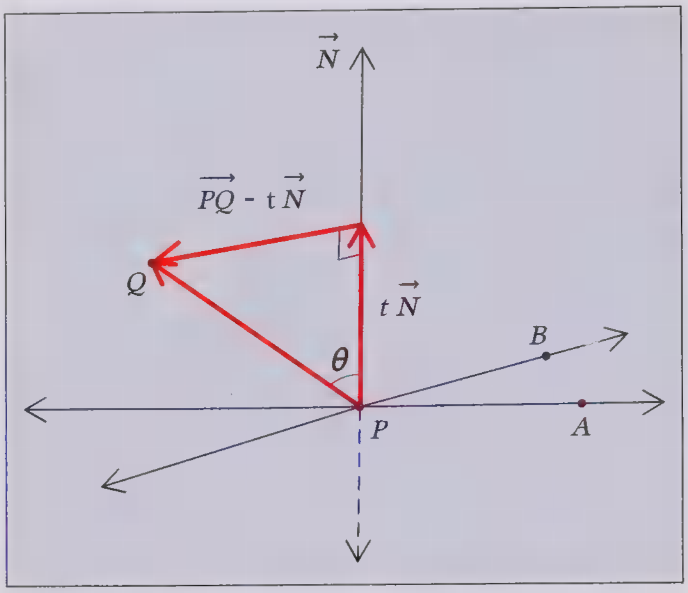


Fig. 2 $\overrightarrow{PQ}$ is projected onto the normal vector $\vec{N}$ from plane *PAB*.

$$t = \frac{\overrightarrow{PQ} \bullet \vec{N}}{\vec{N} \bullet \vec{N}}, \text{ or } t = \frac{\left|\overrightarrow{PQ}\right| \bullet \cos\theta}{\left|\vec{N}\right|}.$$

Note that t has the same sign as $\cos\theta$. So Q is in the half space of π pointed to by $\vec{N}$ if and only if $0° < \theta < 90°$ (i.e., $t > 0$). Likewise, Q is in the opposite half space if and only if $90° < \theta < 180°$ (i.e., $t < 0$).

We observe that $d(Q, \pi) = |t\vec{N}| = |t||\vec{N}|$ and that the signed distance, $SD(Q, \pi)$ relative to $\vec{N}$ is $t|\vec{N}|$, or, equivalently, $|\overrightarrow{PQ}|\cos\theta$. More important, an easy-to-use formula for $SD(Q, \pi)$ is given by

$$SD\left(Q,\pi\right) = \frac{\left|\overrightarrow{PQ}\right| \bullet \left|\vec{N}\right| \bullet \cos\theta}{\left|\vec{N}\right|} = \frac{\overrightarrow{PQ}}{\vec{N}}, \quad (2)$$

which does not require θ or the equation for π.

THEOREM 2

Theorem 2 may be stated as follows:

> For a regular tetrahedron, *PQRS*, the sum of the signed distances from any point to the extended plains of the faces of *PQRS* is a constant and equals the height of *PQRS*.

Proof

There is no loss of generality in letting the edges of the tetrahedron be two units in length and oriented in the *x,y,z*-coordinate system (see **fig. 3**). We will let π_1, π_2, π_3, and π_4 denote the extension planes of the four faces of the tetrahedron. For computing the signed distances to these planes, we need normal vectors to these planes that point to the half space containing the interior of the tetrahedron (see definition 1). Using the right-hand rule for cross products (Briggs and Cochran 2011, pp. 714), we can find these normal vectors by computing as follows:

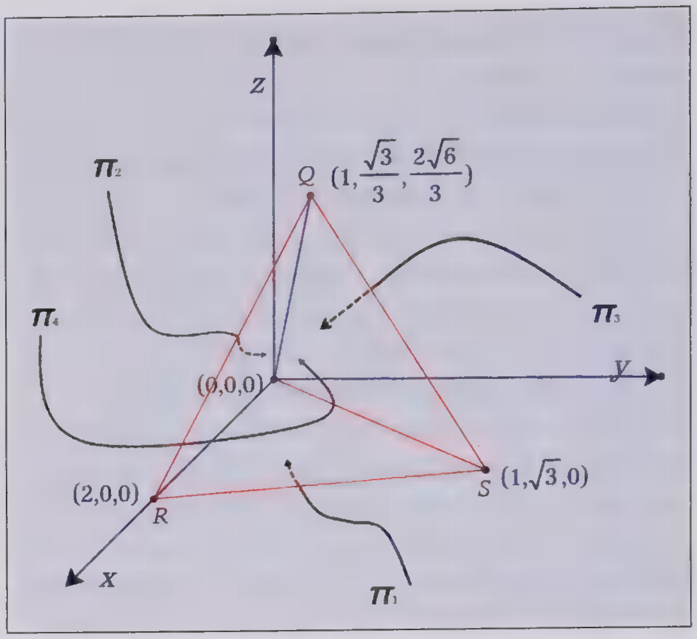


Fig. 3 Students positioned their regular tetrahedron with point *P* at the origin.

$$\overrightarrow{N_1} = \overrightarrow{PR} \times \overrightarrow{PS}$$
$$\overrightarrow{N_2} = \overrightarrow{PR} \times \overrightarrow{PR}$$
$$\overrightarrow{N_3} = \overrightarrow{PS} \times \overrightarrow{PQ}$$
$$\overrightarrow{N_4} = \overrightarrow{SR} \times \overrightarrow{SQ}$$

For example, the normal to π_2 is given by

$$\overrightarrow{N_2} = \overrightarrow{PQ} \times \overrightarrow{PR} = \left\langle 0, \frac{4\sqrt{6}}{3}, -\frac{2\sqrt{3}}{3} \right\rangle.$$

Next, if we use equation (2) for any point $M(p, q, r)$ in three-space, the signed distance from M to the plane π_2 is given by

$$SD\left(M,\pi_2\right) = \frac{\overrightarrow{PM} \bullet \overrightarrow{N_2}}{\left|\overrightarrow{N_2}\right|} = \frac{\langle p,q,r\rangle \bullet \left\langle 0, \frac{4\sqrt{6}}{3}, \frac{-2\sqrt{3}}{3}\right\rangle}{\sqrt{\left(\frac{4\sqrt{6}}{3}\right)^2 + \left(\frac{-2\sqrt{3}}{3}\right)^2}}$$
$$= \frac{2\sqrt{2}q - r}{3}.$$

We pause for a quick illustration of the validity of this formula. Pick a point $M_1 = S$ with coordinates $(1, \sqrt{3}, 0)$, which is in the same half space as the tetrahedron, and M_2 with coordinates $(0, -2, 0)$, which is not. When finding $SD(M, \pi_4)$, we cannot use point P because P does not lie on the plane π_4. We may use point Q, R, or S. We note that $SD(M_1, \pi_2)$ is positive (the height of the tetrahedron) and that $SD(M_2, \pi_2)$ is negative:

$$SD(M_1,\pi_2) = \frac{2\sqrt{2}q - r}{3}\bigg|_{\langle 1,\sqrt{3},0\rangle} = \frac{2\sqrt{6}}{3}$$

and

$$SD(M_2,\pi_2)=\left.\frac{2\sqrt{2}q-r}{3}\right|_{\langle 0,-2,0\rangle}=\frac{-4\sqrt{2}}{3}$$

Returning to the proof, we need the signed distances from a point M to the other three planes: π_1, π_3, and π_4. Using the approach above, we find that these distances are as follows:

$$SD(M,\pi_1)=r$$

$$SD(M,\pi_3)=\frac{\sqrt{6}p-\sqrt{2}q-r}{3}$$

$$SD(M,\pi_4)=\frac{-\sqrt{6}p-\sqrt{2}q-r+2\sqrt{6}}{3}$$

Thus,

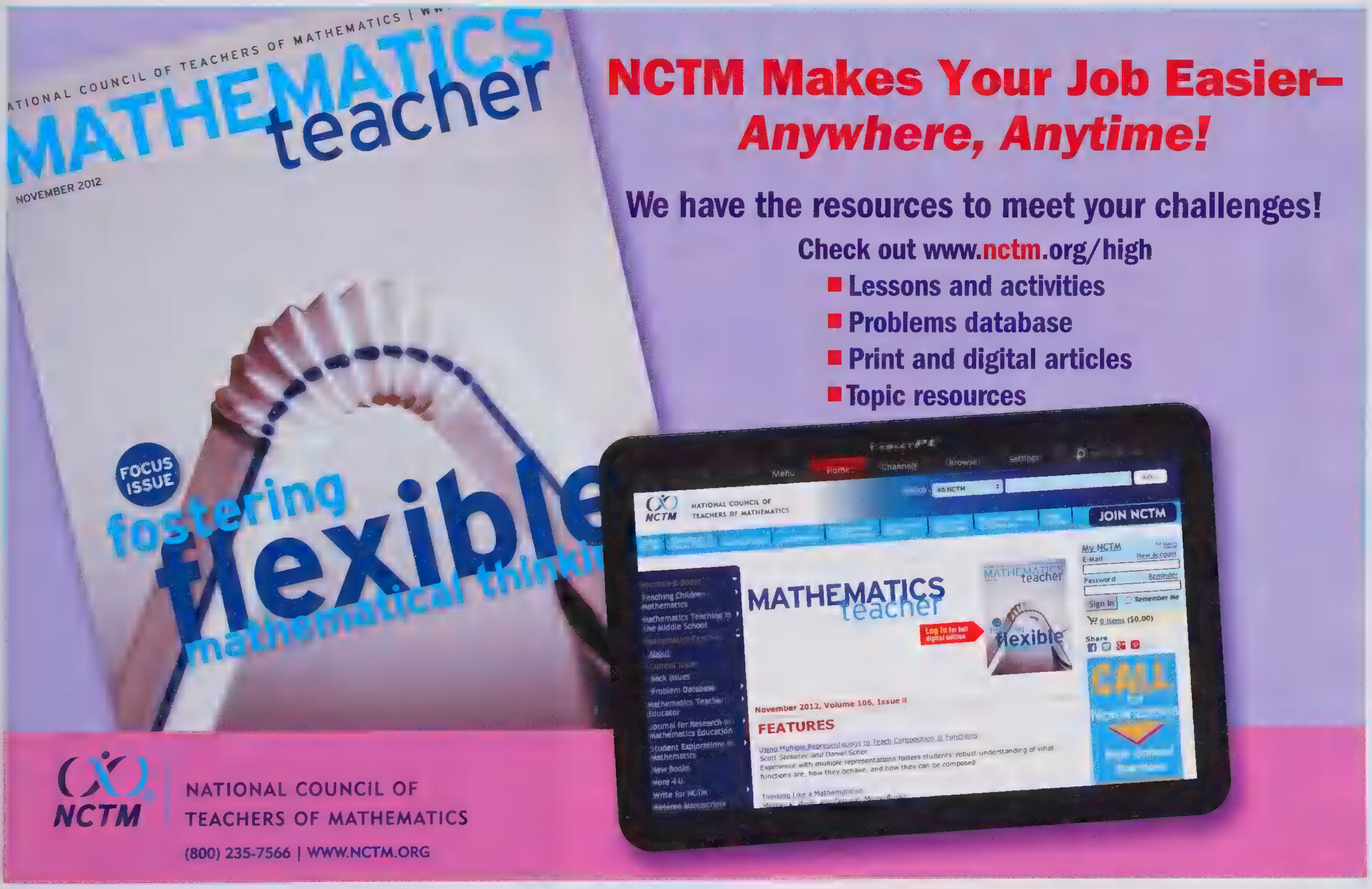
$$\begin{aligned}\sum_{i=1}^{4}SD(M,\pi_i)&=r+\frac{2\sqrt{2}q-r}{3}+\frac{\sqrt{6}p-\sqrt{2}q-r}{3}\\&\quad+\frac{-\sqrt{6}p-\sqrt{2}q-r+2\sqrt{6}}{3}\\&=\frac{2\sqrt{6}}{3},\end{aligned}$$

which is the height of the tetrahedron. This step completes the proof of theorem 2, the three-dimensional extension of Kalish's surfer problem. We can see that theorem 1 is a corollary of theorem 2.

As stated in the Common Core State Standards, "proficient students … try special cases and simpler forms of the original problem in order to gain insight into its solution" (CCSSI 2010, p. 6). Such was the case for the two students who searched for extensions to Kalish's surfer problem. Moreover, on reaching these conclusions, they quickly saw that if they had enclosed their original triangular prism and submerged it in the ocean, they would have obtained similar results. Over the course of this project, these students came to internalize many of the Standards for Mathematical Practice, incorporating them into their own understanding and approaches to mathematics both as learners and as future teachers. Over time, this internalization became clear to their other mathematics and education instructors as well and was evidenced by their increased confidence in classroom and field experiences.

FURTHER INQUIRY AND CREATIVITY

Opportunities abound for additional inquiry and creativity regarding the surfer problem. The reasoning presented here can be applied to all enclosed right regular polygonal prisms, not just the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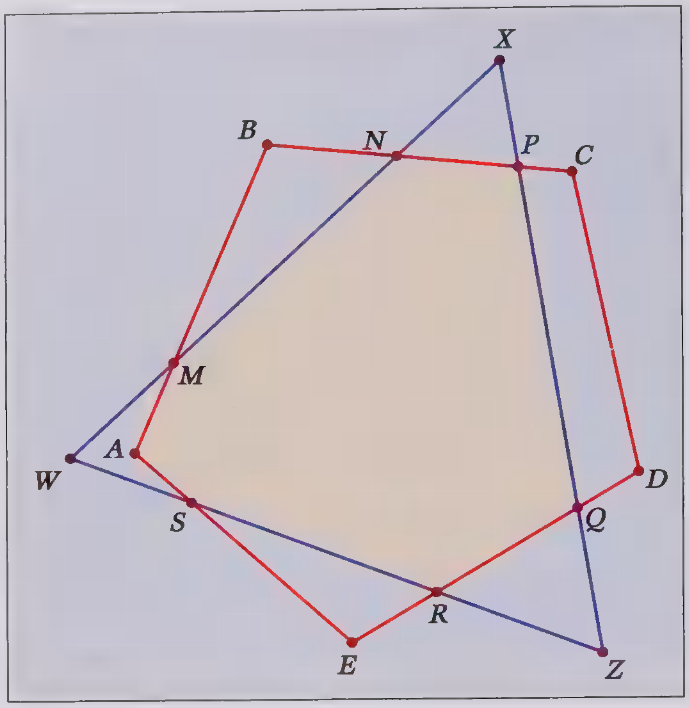

Fig. 4 The intersection of a regular pentagon and an equilateral triangle creates a convex region that can serve as a source for further inquiry.

equilateral triangular prism. Students could explore the possibility of finding a prism whose base is not a regular polygon but produces similar results. Such a discovery would lead them to search for irregular polygons with the property that the sum of the distances of any interior point to the sides is a constant. Figures with rotational symmetries, such as parallelograms, and figures formed by intersections of figures with rotational symmetries would be a rich starting point.

For example, consider the convex region formed by the intersection of a regular pentagon and an equilateral triangle that share a common center point (see **fig. 4**). Students could search for alternative proofs of theorems 1 and 2 or extend these results to other Platonic or Archimedean solids. Also, students might be challenged to define signed distances in higher dimensions and investigate how these results might extend to other dimensions.

WHAT WE CAN LEARN

Using the surfer problem and its three-dimensional analogue, the diver problem, has allowed us to model teaching practices aligned with the ideas set forth in the Standards for Mathematical Practice in the Common Core State Standards (CCSSI 2010). This problem and its extensions could also be used to address many other practices, including group work, proof and reasoning, and technology (interactive geometry software). As teachers of teachers, we hope that all preservice teachers have the opportunity to be fully immersed in the standards. Then they can effectively instill these mathematical practices in their students.

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Priest, Dean B., and Ron Smith. 2010. "Reader Reflections: Extending a Classical Geometry Problem." *Mathematics Teacher* 103 (6): 390–91.

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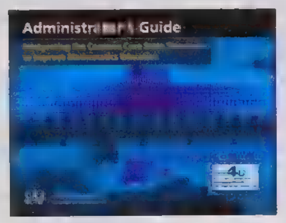
Administrator's Guide: Interpreting the Common Core State Standards to Improve Mathematics Education, Matthew R. Larson, 2011. 113 pp., $23.95 paper. ISBN 978-0-85753-695-0. Stock no. 14288. National Council of Teachers of Mathematics; www.nctm.org.



This administrator's guide provides a compact resource for administrators and leaders of mathematics teachers to support teachers in implementing the Common Core State Standards (CCSS). The author presents the current state of mathematics education and offers a rationale for the CCSS. Central to the book is an explanation of how NCTM publications support better understanding of the CCSS.

According to Larson, three critical NCTM publications—*Principles and Standards for School Mathematics* (2000), *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (2006), and *Focus in High School Mathematics: Reasoning and Sense Making* (2009)—stand as exemplars for clarifying the CCSS. These resources provide a wider lens for viewing and understanding the new curricular expectations in the CCSS. If teachers are already familiar with these readings, then this administrator's guide will be a helpful step in the process of studying and implementing new standards.

This excellent resource allows facilitators the flexibility to adjust the entry point for studying new standards to mathematics teachers' experiences. The key ideas and questions guide teachers' conversations through professional development to professional learning communities.

This book is not one to be read and then put on the bookshelf to gather dust. Rather, it should be revisited repeatedly when trying to understand and implement the CCSS.

—Linda Gann
Northside Independent School District
San Antonio, TX

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FROM OTHER PUBLISHERS

The Glorious Golden Ratio, Alfred S. Posamentier and Ingmar Lehmann, 2011. 363 pp., $27.00 cloth. ISBN 978-1-61614-423-4. Prometheus Books; www.prometheusbooks.com.

This book presents the history of the golden ratio ϕ and its surprising appearances throughout mathematics. It begins by tracing this ratio back to Herotodus's account of the construction of the Great Pyramid of Khufu. Continuing with sixteen geometric constructions of ϕ, the authors discuss the occurrence of the golden rectangle in the art and architecture of antiquity, with appropriate skepticism.

Posamentier and Lehmann illustrate the appearance of the golden ratio in many geometric figures, including pentagons and the Platonic solids. They clearly illuminate the connection between the golden ratio and the Fibonacci sequence as well as its representation as a repeated radical. In a series of twenty-eight "curiosities," the authors detail unexpected numeric, geometric, and algebraic appearances of ϕ. Additional chapters address the relationship between the golden ratio and fractals and the appearance of ϕ in the spiral patterns of plants such as sunflowers and pineapples.

The authors successfully argue that ϕ is ubiquitous throughout mathematics and present their many examples in an informal yet precise manner. This book could be read by anyone with knowledge of high school geometry. Some topics rise above this level of mathematical sophistication, but the mathematics content is presented clearly.

Teachers of geometry or algebra could easily adapt many of the examples to create in-depth projects for the entire class or provide enrichment for students needing additional challenge.

—Chris Black
Central Washington University
Ellensburg, WA

In Pursuit of the Traveling Salesman: Mathematics at the Limits of Computation, William J. Cook, 2012. 272 pp., $27.95 cloth. ISBN 978-0-691-15270-7. Princeton University Press; www.press.princeton.edu.

William J. Cook sets out to explain the history, challenges, and beauty of the Traveling Salesman Problem (TSP), a classic mathematics problem. The early chapters are devoted to its history, including Euler circuits and Hamiltonian circuits. Applications beyond the simple salesman problem, including the movements of telescopes and etching computer chips with lasers, are explored. Later chapters delve into higher-level mathematics that has been developed in the past sixty years to attack the TSP, including subtours, cutting planes, and linear programming. The book concludes with a look at the aesthetics of the problem and its relationship to art.

The first few chapters can be understood by anyone who has taken basic high school mathematics courses. Subsequent chapters would be a challenge for anyone without a formal background in higher-level mathematics, such as graph theory. The author has a solid understanding of the material and tries to present it in an accessible and sometimes entertaining way.

Although this book is at too high a level to be used with many high school students, I recommend it to anyone interested in gaining a deeper understanding of the TSP and modern developments in solving TSP-like problems.

—S. Leigh Nataro
Moravian Academy
Bethlehem, PA

Math for the Frightened: Facing Scary Symbols and Everything Else That Freaks You Out about Mathematics, Colin Pask, 2011. 380 pp., $19.00 paper. ISBN 978-1-61614-421-0. Prometheus Books; www.prometheusbooks.com.

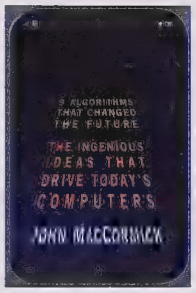
Students are sometimes briefly intrigued by beautiful symmetry and patterns in mathematics but are not motivated to investigate further. *Math for the Frightened* takes some interesting patterns in mathematics and inspires further investigation, building understanding of mathematical concepts. The goal is to show why symbolic representation in mathematics is important, how many mathematical symbols developed, and how they enable the expression of new theories. The intended audience is anyone who is intimidated by the symbolic language of mathematics, who struggled through mathematics courses, and who wants to gain an overview of mathematics development and use in today's world. Educators who are not frightened by mathematics can also benefit from reading the book and may find some interesting ideas to bring to the classroom to capture students' interest.

The mathematical discourse begins with Pythagorean triples and transitions smoothly to a number of interesting patterns with squares of positive integers, developing the idea of generalization and use of mathematical symbols. The author shows that proof is important so that we know that a generalization is always correct. After moving through the development of symbols for mathematical ideas, the author makes connections between algebra and geometry and then moves to the more abstract world of mathematics. Pask brings together many mathematical ideas that may have been seen by many as discrete, unrelated concepts.

Math for the Frightened is an interesting read, not only for those fearful of mathematics but also for those involved in mathematics and the sciences that use mathematics.

—Sandra Nite
Texas A&M University
College Station, TX

Nine Algorithms That Changed the Future: The Ingenious Ideas That Drive Today's Computers, John MacCormick, 2011. 248 pp., $27.95 cloth. ISBN 978-0-691-14714-7. Princeton University Press; www.press.princeton.edu.


9 ALGORITHMS THAT CHANGED THE FUTURE
THE INGENIOUS IDEAS THAT DRIVE TODAY'S COMPUTERS
JOHN MACCORMICK

As a mathematics teacher in a mathematics-computer science department, I enjoyed reading this book on the big ideas of computer science. *Nine Algorithms That Changed the Future* would appeal to those who are in daily contact with computer science students and those who are curious about the details of many daily transactions, such as using a credit card safely on the Internet. The author gives enough detailed mathematical information to interest students at all levels but also has an intriguing way of explaining things for mathematicians and nonmathematicians alike.

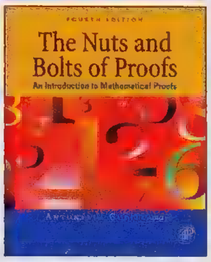
The chapter on public key cryptography is particularly interesting. One analogy the author uses is that coding and decoding numbers is like mixing and unmixing paints. In a delightfully readable manner, MacCormick explains in detail how private information can be transmitted publicly and safely. In addition to the cryptography section, I also enjoyed the discussions on Google, error correcting, and handwriting analysis.

Each section of the book begins with an engaging and informative analogy. Then, once the general knowledge of the topic is conveyed, more mathematical detail is provided for those interested.

I highly recommend this book to anyone—students and teachers of mathematics as well as nonmathematicians who, whether they realize it or not, use the main ideas of computer science every day.

—Anne Quinn
Edinboro University
Edinboro, PA

The Nuts and Bolts of Proofs: An Introduction to Mathematical Proofs, Antonella Cupillari, 2012. 4th ed. 296 pp., $49.95 paper. ISBN 978-0-12-382217-8. Academic Press/ Elsevier; www.elsevier.com.


The Nuts and Bolts of Proofs
An Introduction to Mathematical Proofs

Students who have been successful in algorithmic mathematics and computation often meet their first real mathematical challenge when they are asked to prove something. Fortunately, friendly guides like Cupillari's are available to aid the novice theorem prover.

The Nuts and Bolts of Proof gently leads the reader through the primary techniques of theorem proving. The book begins with a general discussion of the syntax of logical statements, which is followed by an example-driven (mostly number theory) discussion of direct proof and proof by contradiction. Next comes a discussion of "if and only if" theorems, existence and uniqueness theorems, and mathematical induction. The third section of the book (and new to this edition) gives readers an opportunity to apply their new proving skills to theorems from set theory, functions and relations, group theory, and limits. The book concludes with 120 additional exercises (without solutions given) and solutions to the previously encountered exercises.

Cupillari writes in a vibrant conversational style, and her suggestions on approaching and completing proofs are well considered. (The book's few errors seem to be the result of careless editing.) *The Nuts and Bolts of Proof* would serve as a nice supplement to a discrete mathematics course or as a stand-alone book on mathematical proof.

—James V. Rauff
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THANKS FROM THE editorial panel

The success of *Mathematics Teacher* is very much dependent on the volunteer efforts of many mathematics educators. Those who serve as department editors, manuscript referees, and publications and courseware reviewers include middle school and high school teachers, curriculum designers, college and university mathematicians, and teacher educators. Their contributions are deeply appreciated.

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MY FAVORITE lesson

Kenneth A. Horwitz

The Bulldog Lottery

Several years ago I became the lead teacher for a newly created statistics course. I wanted the course to emphasize projects that would allow students to apply statistical knowledge to real-world situations. My Bulldog Lottery lesson allows students to apply the concepts of probability and expected value to a real-world phenomenon.

At the beginning of the lesson, as students enter the room, they are given a lottery ticket for Bulldog Pick 1. This lottery ticket invites purchasers to select one number from the first fifty-six natural numbers.

The first two questions I ask students are these:

- How many different ways are there to choose a number?
- What is the probability that you will win the Bulldog Pick 1 jackpot?

Students do not find these questions difficult, and we quickly move to Bulldog Pick 2, whose prize is \$500 for picking two numbers from the same fifty-six natural numbers. I ask the same two questions as for Bulldog Pick 1, but I extend the dialogue by asking how many different tickets need to be sold to guarantee a winner and still ensure that the state will make a profit from the lottery sales.

I now have an opportunity to introduce the concept of expected value. Using the Bulldog Pick 2 lottery, I ask:

> Suppose there is just one first prize of \$500 for winning the Pick 2 lottery. If 1540 \$1-tickets are sold, what is the expected value if we purchase 1 ticket? This expected value can be computed as $E(x) = 500 \cdot (1/1540) - 1 = -\$0.67$.

I am able to explain the idea of a "fair game" (expected value of 0) and a game that is skewed toward the player (positive expected value). In this case, the –\$0.67 is the amount that a purchaser expects to lose in the long run if he or she keeps buying tickets.

Next, working in small groups, students examine the typical Pick 3 and Pick 5 lotteries. They discuss how many tickets must be sold to advertise a winning ticket periodically while maintaining a game that remains skewed toward the state. After students calculate how many tickets must be sold to guarantee a winner and the expected value of each game, we discuss how the game changes if the numbers must be picked in a specific order.

Then I present each group with a new lottery ticket for the Mega Millions® lottery, which requires selecting five numbers from fifty-six and one Powerball number from among the first forty natural numbers. My students consider the probabilities associated with Mega Millions:

- What is the probability of matching all five numbers?
- What is the probability of picking the Powerball number?
- What is the probability of picking the five regular numbers *and* the Powerball number?

Afterward, students create their own lottery. They are encouraged to be original while maintaining a sense of realism about where and how tickets will be distributed. The game must have a probability of winning between that of the Pick 5 lottery and that of the Mega Millions lottery. Students present their lottery, including a ticket, probability calculations, and other information. As an exercise to gain state approval of their lottery, they must propose a use for the money that will benefit the public.

Over the years, the original lesson has been expanded to extend over several days. Small groups are encouraged to work together, and the extra time allows the more creative people in the class a chance to excel. In this way, the mathematics comes alive for everyone in the room.

The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

Edited by ***Jennifer Wexler,*** *wexlerj@newtrier.k12.il.us, New Trier High School, Winnetka, IL*

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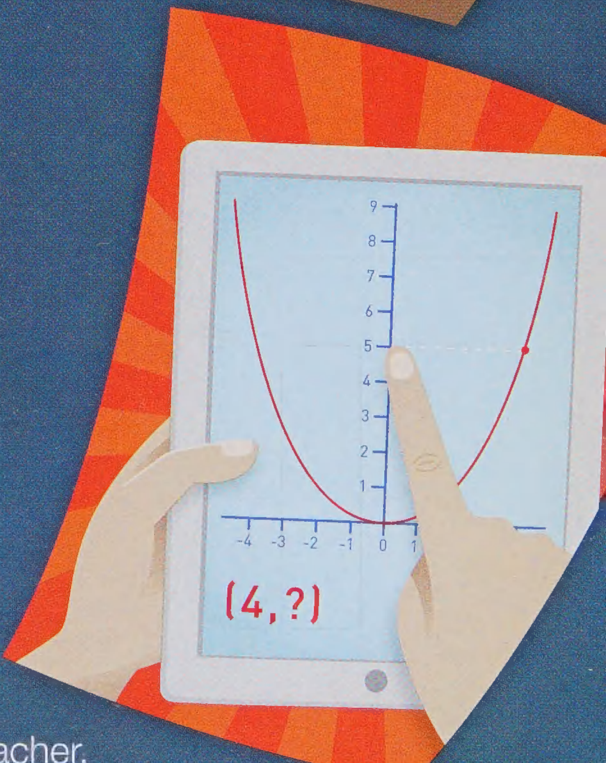

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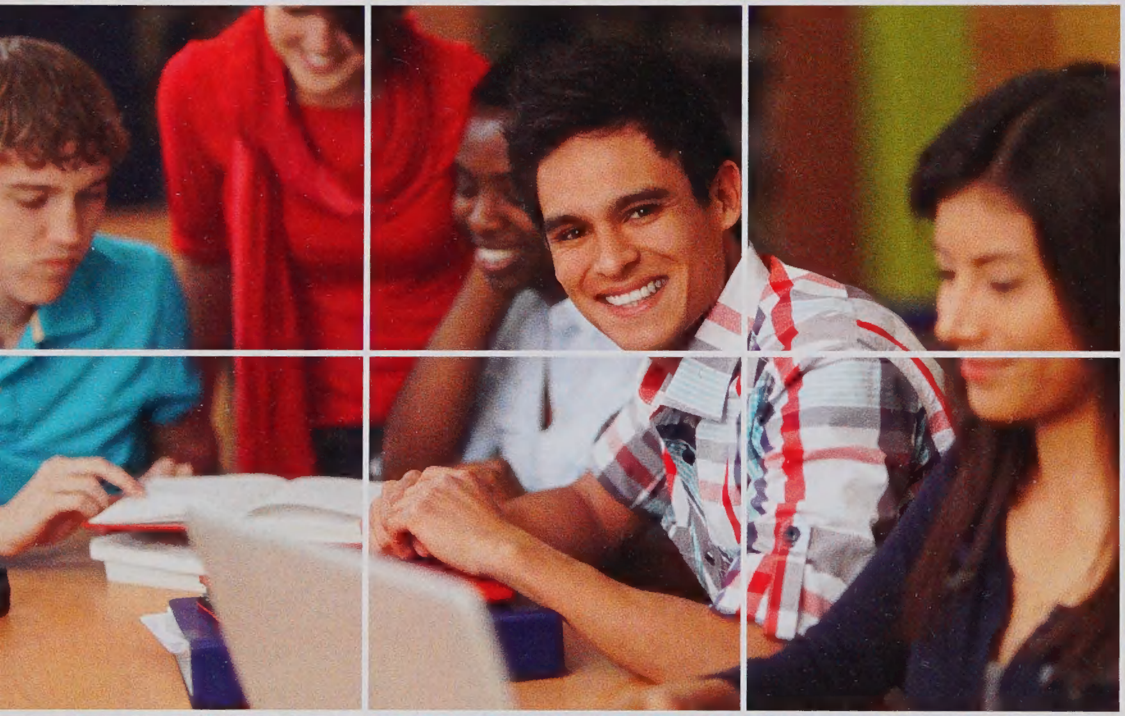

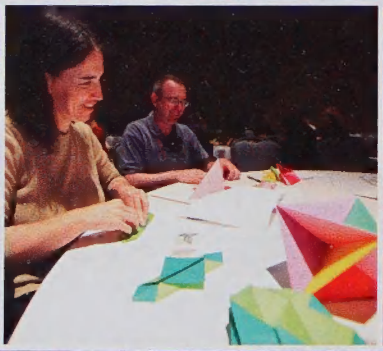

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